Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

[Exercise 5442]

Chain of coupled rotors

A one-degree-of-freedom rotor is described by the Hamiltonian $p^2/(2\mathsf{m}a^2)$ where $\mathsf{m}a^2$ is the moment of inertia. Its interaction with an external field is described by $-h\cos(\varphi)$. The magnetization is $M \equiv \langle\cos(\varphi)\rangle$. In thermal equilibrium at temperature T, it can be expressed as $M = R(\chi h)$ where $R()$ is some nonlinear function that is defined such that $R'(0) = 1$, and χ is the susceptibility,

We consider a chain of such rotors. Near neighbor rotors are coupled by the interaction $-\varepsilon\cos(\phi_i)$, where $\phi_i = \varphi_i - \varphi_{i-1}$. We define the correlation function $C(r) = \langle\cos(\varphi_r)\cos(\varphi_0)\rangle$.

(1) For a single classical rotor at temperature T determine the susceptibility χ. Additionally write what is the function $R(x)$. The latter expression can be written in terms of definite integrals. Care about the $R'(0) = 1$ convention.

(2) For a chain of classical rotors, determined what is T_c and what is χ for $T > T_c$ on the basis of mean field theory.

(3) For a chain of classical rotors, calculate the correlation function $C(r)$, and the implied correlation scale ξ. Express the result using the function $R()$.

(4) Determine what is χ on the basis of an exact calculation. Express your result in terms of ξ.

(5) Considering a single quantum rotor at $T = 0$, use dimensional analysis in order to guess what is χ. If an actual calculation is done the numerical prefactor is 2.

(6) For a chain of quantum rotors, determine, via same mean-field reasoning as in item 2, the condition on $\mathsf{m}a^2$ for observing a phase transition. Additionally, plot a schematic $(\mathsf{m}a^2, T)$ phase diagram with the following lines: phase transition according to mean field theory (dotted); phase transition according to the exact analysis (solid).