Ex4554: Fermions in magnetic field, quantum phase transition

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The problem:

A d dimensional container ($d = 1, 2, 3$) contains fermions of density n, temperature $T = 0$, mass m and spin $\frac{1}{2}$, having a magnetic moment $\bar{m}$. The container is placed in a magnetic field $H/\bar{m}$ so that the fermion spectra is $\epsilon_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} \pm H$ where $\mathbf{p}$ is the momentum. (Note that orbital effects are neglected, possible e.g. at d=2 with the field parallel to the layer).

(a) Evaluate the chemical potential $\mu(H)$, for small H: Consider first an expansion to lowest order in H and then evaluate $d\mu/dH$ to note the change at finite H.

(b) Beyond which H_c does the consideration in (a) fail? Find $\mu(H)$ at $H > H_c$ and plot qualitatively $\mu(H)/\mu_0$ as function of H/μ_0 (where $\mu_0 = \mu(H = 0)$) for $d = 1, 2, 3$, indicating the values of $\mu(H)/\mu_0$ at H_c.

(c) Of what order is the phase transition at H_c, at either $d = 1, 2, 3$? Does the phase transition survive at finite T? (no need for finite T calculations – just note analytic properties of thermodynamic functions).

(d) The container above, called A, with $H \neq 0$ is now attached to an identical container B (same fermions at density n, $T = 0$), but with $H = 0$. In which direction will the fermions flow initially? Specify your answer for $d = 1, 2, 3$ at relevant ranges of H.

The solution:

(a) To find μ, first we find the density of states $g(\epsilon)$, then the total number of particles $N(\mu, H)$, and invert the equation:

$g(\epsilon)$ gets a separate contribution from each spin:

$$g(\epsilon) = g_+(\epsilon) + g_-(\epsilon) = g_f(\epsilon - H) + g_f(\epsilon + H)$$

Where $g_f(\epsilon) = Vc\epsilon^{\frac{d}{2}-1}$ is standard free-particle energy density. Here c is the volume inside the unit d-sphere times $\frac{d}{2}\left(\frac{\sqrt{2m}}{2\pi}\right)^{d/2}$.

N is given by:

$$N = \int g(\epsilon) f(\epsilon - \mu) d\epsilon$$

Where f at $T = 0$ is just a step function limiting integration up to $\epsilon_F = \mu$. So

$$N = \int_{-\infty}^{\mu} g_f(\epsilon - H) + g_f(\epsilon + H) d\epsilon$$

$$= \frac{2Vc}{d}\left[\epsilon^{d/2}|_0^{\mu+H} + \epsilon^{d/2}|_0^{\mu-H}\right]$$

$$= \frac{2Vc}{d}\left[(\mu + H)^{d/2} + (\mu - H)^{d/2}\right]$$

This gives $\mu_0 = \mu(H=0) = \left(\frac{Nd}{4Vc}\right)^{2/d} = \left(\frac{nd}{4c}\right)^{2/d}$.

To find $\mu(H)$ for small H we expand to second order of H and apply$\left(\frac{d}{dH}\right)_{n,T}$

$$N = \frac{2Vc}{d} 2\left[\mu^{d/2} + \frac{d}{2}\left(\frac{d}{2}-1\right)\mu^{\frac{d}{2}-2}H^2 + O(H^3)\right]$$

$$0 = \frac{d}{2}\mu^{\frac{d}{2}-1}\frac{d\mu}{dH} + \frac{d}{2}\left(\frac{d}{2}-1\right)\left(\frac{d}{2}-2\right)\mu^{\frac{d}{2}-3}H^2$$

Taking $H \to 0$, $d\mu/dH = 0$. This can also be seen from the symmetry of the system to $H \to -H$. We can search for μ to second order in H: $\mu(H) = \mu_0 + \mu_2 H^2$

$$N = \frac{2Vc}{d}\left[(\mu_0 + \mu_2 H^2 + H)^{d/2} + (\mu_0 + \mu_2 H^2 - H)^{d/2}\right]$$

Expanding in the small parameters $\mu_2 H^2 + H$, $\mu_2 H^2 - H$

$$= \frac{2Vc}{d} 2\left[\mu_0^{d/2} + \frac{d}{2}\mu_0^{\frac{d}{2}-1}\mu_2 H^2 + \frac{d}{2}\left(\frac{d}{2}-1\right)\mu_0^{\frac{d}{2}-2}H^2 + O(H^3)\right]$$

From comparing H^2 terms we get

$$\mu_2 = -\mu_0^{-1}\left(\frac{d}{2}-1\right)$$

So μ is increases with H for $d=1$, is constant for $d=2$, and decreases for $d \geq 3$. Notice that for $d=2$, $\mu(H)$ is constant for $H < H_c$ (see (b)) with no approximations.

(b) At arbitrary H, we can no longer assume both spin states contribute: the negative energy spin $(-H)$ states might include more then N states with energy lower than the ground state for positive energy spin $(+H)$.

The critical H is therefore when the energy levels up to H in the $(-H)$ spins are equal to N:

$$N = \int_{-\infty}^{H} g_f(\epsilon - H)d\epsilon = \frac{2Vc}{d}\left[\epsilon^{d/2}|_0^{2H}\right] = \frac{2Vc}{d}(2H)^{d/2}$$

$$H_c = \frac{1}{2}\left(\frac{nd}{2c}\right)^{2/d} = \frac{1}{2}\,2^{2/d}\,\mu_0$$

$\mu(H)$ above H_c is given by

$$N = \int_{-\infty}^{\mu} g_f(\epsilon - H)d\epsilon = \frac{2Vc}{d}\left[\epsilon^{d/2}|_0^{\mu+H}\right] = \frac{2Vc}{d}(\mu + H)^{d/2}$$

$$\mu(H) = \left(\frac{nd}{2c}\right)^{2/d} - H = 2^{2/d}\,\mu_0 - H$$

Notice that μ must be continuous in H_c, as for H infinitesimally smaller than H_c the contribution of $g_f(\epsilon+H)$ to the equation is also infinitesimal $(O(H_c - H))$ for smooth $g_f(\epsilon)$. In fact, for $d \geq 3$ we have $g_f(0) = 0$ and the contribution is $O\left((H_c - H)^2\right)$ thus giving a continuous derivative of μ at H_c.

Let us prove this formally.

μ is continuous at H_c:

Consider $\delta H > 0$. The density is constant and so is equal calculating at $H^- = H_c - \delta H$ and $H^+ = H_c + \delta H$.

$$n = \frac{2c}{d}\left[(\mu^- + H^-)^{d/2} + (\mu^- - H^-)^{d/2}\right] = \frac{2c}{d}(\mu^+ + H^+)^{d/2}$$

We already know $\mu^+(H_c) = H_c$ so taking δH to 0 gives

$$(\mu^- + H_c)^{d/2} + (\mu^- - H_c)^{d/2} = (2H_c)^{d/2}$$

Which is solved by $\mu^- = H_c$

μ is differentiable at H_c for $d \geq 3$:

Taking the equation for n for $H < H_c$

$$\frac{d}{dH}n = \frac{2c}{d}\frac{d}{dH}\left[(\mu + H)^{d/2} + (\mu - H)^{d/2}\right] = 0$$

$$\frac{d}{dH}(\mu + H)^{d/2} = -\frac{d}{dH}(\mu - H)^{d/2}$$

$$(\mu + H)^{d/2-1}(\frac{d\mu}{dH} + 1) = -(\mu - H)^{d/2-1}(\frac{d\mu}{dH} - 1)$$

We know in the limit $H \to H_c^-$, $\mu \to H_c$ and with $d \geq 3$, $(\mu - H)^{d/2-1} \to 0$. So

$$0 = \lim_{H \to H_c^-}(\mu + H)^{d/2-1}(\frac{d\mu}{dH} + 1) = (2H_c)^{d/2-1}\lim_{H \to H_c^-}(\frac{d\mu}{dH} + 1)$$

$$\frac{d\mu}{dH}(H_c^-) = -1$$

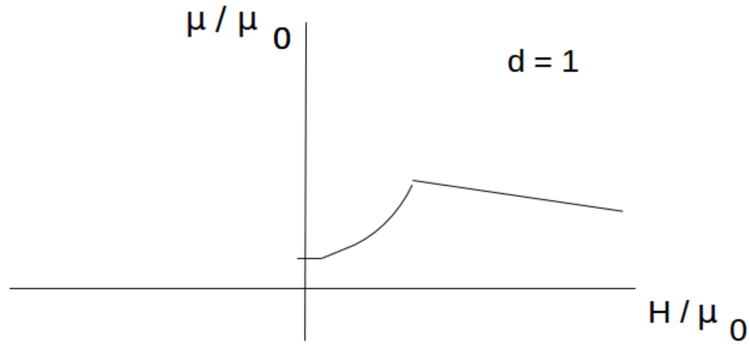
So $\frac{d\mu}{dH}(H_c^-) = \frac{d\mu}{dH}(H_c^+) = -1$ and μ is differentiable at H_c. This also shows that for $d = 1$, $\lim_{H \to H_c^-}(\frac{d\mu}{dH} - 1) = 0$ to negate the "infinite" $\lim_{H \to H_c^-}(\mu - H)^{d/2-1}$.

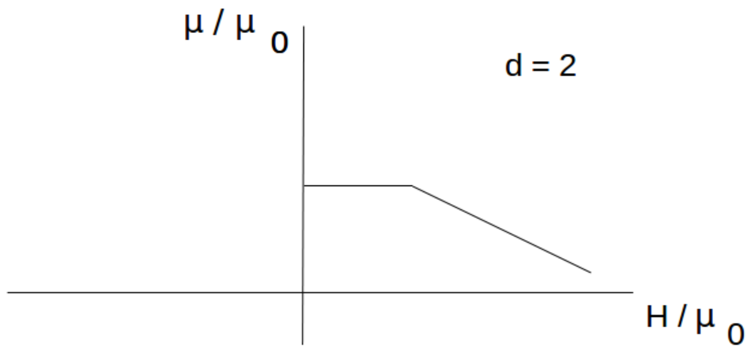
Using the same method, for $d = 3$ the second derivative is not continuous:

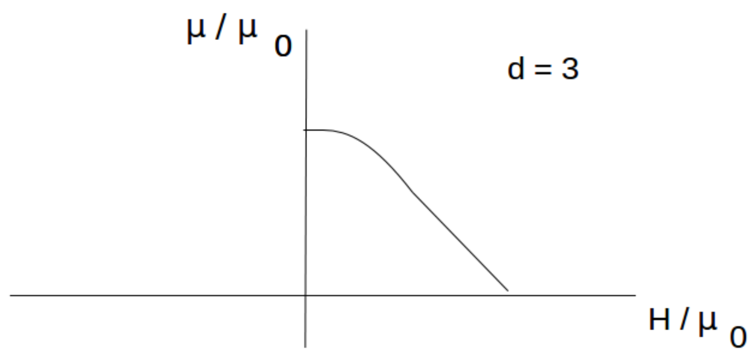
$$(\mu+H)^{d/2-2}\left(\frac{d\mu}{dH} + 1\right)^2 + (\mu+H)^{d/2-1}\frac{d^2\mu}{dH^2} = -(\mu-H)^{d/2-2}\left(\frac{d\mu}{dH} - 1\right)^2 - (\mu-H)^{d/2-1}\frac{d\mu}{dH}$$

$$\lim_{H \to H_c^-}(2H_c)^{d/2-1}\frac{d^2\mu}{dH^2} = \lim_{H \to H_c^-} -4(\mu - H)^{d/2-2} = -\infty$$

Graphs:







(c) We can look at the magnetization at different values of H. For $H > H_c$, all spins are in one direction and magnetization is constant $M = N$. Otherwise

$$M = \int_{-\infty}^{\mu} g_f(\epsilon + H) - g_f(\epsilon - H) d\epsilon = \frac{2Vc}{d}\left[(\mu - H)^{d/2} - (\mu + H)^{d/2}\right]$$

The susceptibility is

$$\chi = \frac{d}{dH} M = \frac{d}{dH}\frac{2Vc}{d}\left[(\mu - H)^{d/2} - (\mu + H)^{d/2}\right]$$

So χ has the same continuity properties as $\frac{d}{dH}\mu$ at H_c. For $d = 1, 2$ the phase transition is of first order (discontinuity), and of second order for $d = 3$ (derivative discontinuity).

At finite temperature, $f(\epsilon-\mu)$ is smoothed and has a positive finite value at $\epsilon > H$ for all finite H, so the transition from one dominant spin to both spins dominant is smooth. Both "sites" always have non-zero occupation. The phase transition does not survive.

(d) Particles will initially flow from higher chemical potential to lower

For $d = 3$:

For all H, $\mu(0) > \mu(H)$ so particles will flow from B to A

For $d = 2$:

If $H \leq H_c$, then the chemical potentials are the same and there will be no flow. Otherwise $\mu(0) > \mu(H)$ so particles will flow from B to A.

For $d = 1$:

If $H \leq H_c$, $\mu(0) < \mu(H)$ and particles will flow from A to B. Otherwise, there exists $H_1 = 3\mu_0$ where $\mu_0 = \mu(H_1)$. For $H \in (H_c, H_1)$, $\mu(0) < \mu(H)$ and particles will flow from A to B. For $H \in (H_1, \infty)$, $\mu(0) > \mu(H)$ and particles will flow from B to A.