E4550: Chemical potential: reaction in ideal quantum gases

Submitted by: Shay Izbitski

The problem:

N fermions of type F that have spin $\frac{1}{2}$ are placed in a volume V. they can decay into a boson of type B that has spin 0 and a fermion of type A that has spin $\frac{1}{2}$. the reaction is $F \to A + B$ and it has an energy gain of ϵ_0 (i.e. $A + B$ have lower energy than F). the masses are m_F, m_A, m_B respectively.

(1) Assuming ideal gases at temperature T, write the equations which determine the densities n_F, n_A, n_B in equilibrium.

(2) Write the equations of (1) at $T = 0$ and describe the densities as functions of ϵ_0. Find ϵ_c such that for $\epsilon_0 > \epsilon_c$ the number of F fermions vanishes.

(3) Assume that the condensation of bosons B occurs at T_c such that $T_c \ll \frac{p_F^2}{2m_A} < \epsilon_0$ where p_F is the Fermi momentum of fermions A. Evaluate T_c and rewrite the condition $T_c \ll \frac{p_F^2}{2m_A}$ in terms of the given parameters.

The solution:

(1) The particles are fermions and bosons so clearly there densities are (assuming there is no degeneracy in the boson gas):

$$n_F = 2\sum_p \frac{1}{e^{\beta(\frac{p^2}{2m_F} - \mu_F)} + 1}$$

$$n_A = 2\sum_p \frac{1}{e^{\beta(\frac{p^2}{2m_A} - \mu_A)} + 1}$$

$$n_B = \sum_p \frac{1}{e^{\beta(\frac{p^2}{2m_B} - \mu_B)} - 1} + n_B^0$$

Where n_B^0 is the bose condensation related term.

(2) At $T = 0$ the bosons undergo a condensation meaning that

$$n_B = n_B^0$$

$$\mu_B = 0$$

The chemical potential must obey the reaction by

$$\mu_F + \epsilon_0 = \mu_A + \mu_B$$

and at $T = 0$

$$\mu_A = \mu_F + \epsilon_0$$

in addition, the fermionic density must obey the reaction by

$$n = n_A + n_F = const$$

therefore at $T = 0$ the fermionic density is

$$n = \frac{1}{3\pi^2\hbar^3}(2m_F\mu_F)^{\frac{3}{2}} + \frac{1}{3\pi^2\hbar^3}(2m_A\mu_A)^{\frac{3}{2}}$$

From all of the above it's clear that the condition for $n_F = 0$ is $\mu_F < 0$ leading to

$$0 = \mu_A - \epsilon_c \to n = \frac{1}{3\pi^2\hbar^3}(2m_A\epsilon_c)^{\frac{3}{2}}$$

The density $\frac{n_F}{n}$ starts from the value $\frac{n_F}{n} = \frac{m_F^{\frac{3}{2}}}{m_F^{\frac{3}{2}}+m_A^{\frac{3}{2}}}$ at $\epsilon_0 = 0$ and decrease until it vanishes at $\epsilon_0 = \epsilon_c$.

$n_A = n - n_F$ and therefore the density $\frac{n_A}{n}$ starts from the value $\frac{n_A}{n} = \frac{m_A^{\frac{3}{2}}}{m_F^{\frac{3}{2}}+m_A^{\frac{3}{2}}}$ at $\epsilon_0 = 0$ and increase until it gets the constant value 1 at $\epsilon_0 = \epsilon_c$.

(3) Under the given assumption of $T_c \ll \frac{p_F^2}{2m_A} < \epsilon_0$ and using the fact that $\frac{p_F^2}{2m_A} = \mu_F + \epsilon_0$ at $T = 0$ we get that

$$\frac{p_F^2}{2m_A} = \mu_F + \epsilon_0 < \epsilon_0 \to \mu_F < 0$$

$\mu_F < 0$ leads us to $n_F = 0$, leading to $n_A = n_B = n$ which eventually leads us to

$$\mu_F + \epsilon_0 = \frac{(3\pi^2\hbar^3 n)^{\frac{2}{3}}}{2m_A}$$

putting that back in the condition we are given gives us

$$T_c \ll \frac{(3\pi^2\hbar^3 n)^{\frac{2}{3}}}{2m_A}$$

From the other hand it's known that since $n_B = n$

$$T_c = \frac{2\pi\hbar^2 n^{\frac{2}{3}}}{m_B(2.612)^{\frac{2}{3}}}$$

substituting this to the former condition we get the condition in the given parameters

$$\frac{m_B}{m_A} \ll \frac{4\pi}{(3\pi^2 \cdot 2.612)^{\frac{2}{3}}} = 0.69$$