Ex4546: Chemical equilibrium for misc reactions

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The problem:

(a) The following reaction occurs inside a star

$$\gamma + \gamma \leftrightarrow e^+ + e^-$$

where γ is a photon and $e^\pm$ are the positron and electron, respectively. Assume overall charge neutrality and that the system is in equilibrium at temperature T. Find an expression for the densities of $e^\pm$. (In general $e^\pm$ with mass m are relativistic). Find an explicit result for these densities in the limit $k_BT << mc^2$. (Hint: no conservation law for photons).

(b) Repeat (a) for the reaction

$$\gamma + \gamma \leftrightarrow \pi^+ + \pi^-$$

where $\pi^\pm$ are bosons with mass m_π. Can these bosons become Bose-condensed if the temperature is sufficiently lowered? What are the densities at $T = 0$?

(c) The photons in (b) are now eliminated from the system, but the density of the bosons is maintained. If these bosons are cooled, is there a temperature at which Bose condensation is possible? [Assume that the initial temperature satisfies $k_BT << m_\pi c^2$]. What are the densities at $T = 0$?

(d) Consider initial equal densities $\bar{n}$ for e^+ and for e^-. The $e^\pm$ are allowed to equilibrate with $\pi^\pm$ via the reaction

$$e^+ + e^- \leftrightarrow \pi^+ + \pi^-$$

Write equations that determine the chemical potentials at $T \neq 0$. Consider now $T = 0$ and determine if Bose Einstein condensation is possible and if so what is the condition for that. Consider both $m_e > m_\pi$ and $m_e < m_\pi$ situations.

The solution:

(a) The basic equation which relates the different components in a chemical reaction is the chemical equilibrium equation which states for the desired reaction that,

$$\mu_+ + \mu_- = \mu_\gamma + \mu_\gamma \tag{1}$$

Since the photons have zero rest mass they don't have a conservation law, which means $\mu_\gamma = 0$, making the equation $\mu_+ + \mu_- = 0$. Moreover, charge neutrality dictates that $n_+(\mu_+) = n_-(\mu_-)$ and since both particles (electrons and positrons) have the same energy dispersion and spin, the function n_+ and n_- will be the same with respect to their arguments, which results in having the same chemical potential $\mu_+ = \mu_-$. Combining both equations we have $\mu = \mu_- = \mu_+ = 0$. In the case of relativistic fermions, n as a function of μ is not analytic (not given by elementary functions). The integral expression would be

$$n_+ = n_- = n = \int_m^\infty \frac{g(\varepsilon)/V}{e^{\beta(\varepsilon-\mu)}+1} d\varepsilon = \frac{1}{\pi^2}\int_m^\infty \frac{\varepsilon\sqrt{\varepsilon^2 - m^2}}{e^{\beta\varepsilon}+1} d\varepsilon \tag{2}$$

Where $\mu = 0$ was used and the density of states was derived from the dispersion $\varepsilon = \sqrt{k^2 + m^2}$ (we use natural units $\hbar = c = 1$):
We have

$$\frac{dk}{d\varepsilon} = \frac{d}{d\varepsilon}\left(\sqrt{\varepsilon^2 - m^2}\right) = \frac{\varepsilon}{\sqrt{\varepsilon^2 - m^2}}$$

The number of states between $k, k + dk$ is

$$\frac{g\left(\varepsilon\right) d\varepsilon}{V} = \frac{2d^3k}{\left(2\pi\right)^3} = \frac{8\pi k^2 dk}{\left(2\pi\right)^3} = \frac{1}{\pi^2}\left(\varepsilon^2 - m^2\right)\frac{dk}{d\varepsilon}d\varepsilon = \frac{1}{\pi^2}\left(\varepsilon^2 - m^2\right)\frac{\varepsilon}{\sqrt{\varepsilon^2 - m^2}}d\varepsilon = \frac{1}{\pi^2}\varepsilon\sqrt{\varepsilon^2 - m^2}d\varepsilon \tag{3}$$

Where the factor 2 in the first term is for spin degeneracy.

The limit $T \ll m$ is now taken, leading to a Boltzmann approximation to the occupation function since $e^{\beta\varepsilon} \gg 1$. Thus, the integral would be:

$$\begin{aligned} &n_+ = n_- = n = \tfrac{1}{\pi^2}\int\limits_m^\infty \varepsilon\sqrt{\varepsilon^2 - m^2}e^{-\beta\varepsilon}d\varepsilon = T\left(\tfrac{m}{\pi}\right)^2 K_2\left(\tfrac{m}{T}\right) \approx \\ &\approx e^{-\beta m}\tfrac{m^3}{\pi^2}\sqrt{\tfrac{\pi}{2}}\left(\tfrac{1}{\beta m}\right)^{3/2} == 2\tfrac{(2m)^{3/2}}{(2\pi)^2}\Gamma\left(\tfrac{3}{2}\right)T^{3/2}e^{-\beta m} \end{aligned} \tag{4}$$

Where $K_2(x)$ is the modified Bessel function of the second kind.

We can see that in the limit $T = 0$ the density of the electrons/positrons is 0, thus all the particles annihilate to make a photon gas.

(b) For the boson gas the integral for the number of particles with relativistic energy would be:

$$\begin{aligned} &n_+^{>0} = n_-^{>0} = n^{>0} = \tfrac{1}{2\pi^2}\int \tfrac{\varepsilon\sqrt{\varepsilon^2-m^2}}{e^{\beta(\varepsilon-\mu)}-1}d\varepsilon \approx \tfrac{1}{2\pi^2}\int\limits_m^\infty \varepsilon\sqrt{\varepsilon^2 - m^2}e^{-\beta\varepsilon}d\varepsilon = \\ &= \tfrac{T}{2}\left(\tfrac{m}{\pi}\right)^2 K_2\left(\tfrac{m}{T}\right) \approx \tfrac{(2m)^{3/2}}{(2\pi)^2}\Gamma\left(\tfrac{3}{2}\right)T^{3/2}e^{-\beta m} \end{aligned} \tag{5}$$

Where:
1. Like item (a) the chemical potentials obey $\mu = \mu_- = \mu_+ = 0$ since $\mu_\gamma = 0$.
2. The same limit as in item (a) - $T \ll m$ is taken.
3. $g(\varepsilon)$ is obtained exactly the same as in item (a) except for the spin degeneracy, meaning:

$$\frac{g\left(\varepsilon\right) d\varepsilon}{V} = \frac{1}{2\pi^2}\varepsilon\sqrt{\varepsilon^2 - m^2}d\varepsilon$$

Thus, we can see that at the limit $T \to 0$ the number of excited bosons goes to zero. In addition since the chemical potential is always zero, it is lower than the minimum energy $\mu = 0 < m = \varepsilon_{min}$ forbidding a condensation to appear. Hence, taking the limit of $T \to 0$ we get that all the pions have been annihilated to make a photon gas.

(c) When the photons are eliminated, there is no chemical equilibrium and the pions regain their maximum chemical potential $\mu_{max} = \varepsilon_{min} = m_\pi$, and the gas becomes a regular three-dimensional ideal gas which can condensate. Quantitatively for $\mu \to \varepsilon_{min} = m_\pi$,

$$n_+^{>0} = n_-^{>0} = n^{>0} = \frac{1}{2\pi^2}\int\limits_m^\infty \frac{\varepsilon\sqrt{\varepsilon^2 - m^2}}{e^{\beta(\varepsilon - m)} - 1}d\varepsilon \tag{6}$$

Specifically at the limit $T \to 0$ The integrand is exactly 0, so there are no excited particles and all the pions occupy the ground state leading to $n_+ = n_- = n = \frac{N}{V}$ where $\frac{N}{V}$ is the density of the bosons which was maintained after the elimination of the photons from the system.

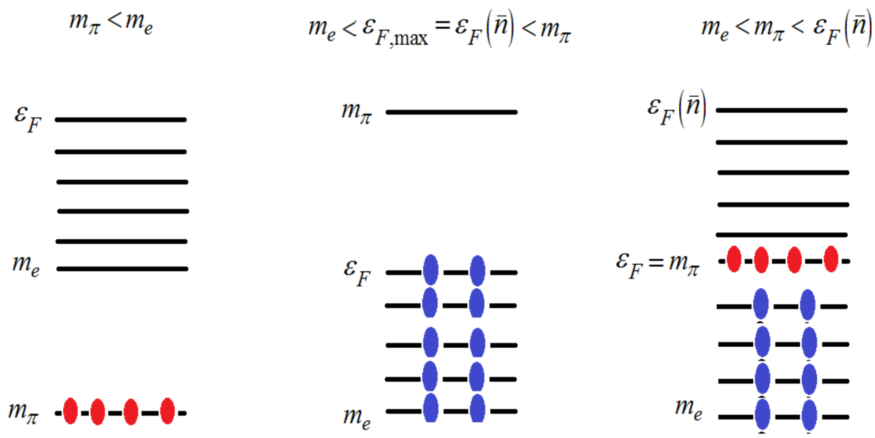

Figure 1: Energy levels and occupation for different mass relation and overall number of particles. The red particles indicate bosons as a BEC, the blue ones indicate fermions.

(d) Since the initial densities are equal the reaction enforces

$$n_{e+} = n_{e-} \qquad n_{\pi+} = n_{\pi-} \qquad n_{e+} + n_{\pi+} = \bar{n} \tag{7}$$

and for the chemical potentials themselves: $\mu_{e+} + \mu_{e-} = \mu_{\pi+} + \mu_{\pi-}$. At $T = 0$ we would have $\varepsilon_{F+} + \varepsilon_{F-} = \mu_{\pi+} + \mu_{\pi-}$. Again, the dispersion for e^+ and π^+ is the same as e^- and π^- respectively so the relation is simplified to $\varepsilon_F = \mu_\pi$
. The density of the fermions is given by:

$$n_e = \frac{1}{\pi^2}\int_{m_e}^{\infty} \frac{\varepsilon\sqrt{\varepsilon^2 - m_e^2}}{e^{\beta(\varepsilon-\mu_e)}+1}d\varepsilon \to \frac{1}{\pi^2}\int_{m_e}^{\infty}\Theta\left(\varepsilon_F - \varepsilon\right)\varepsilon\sqrt{\varepsilon^2 - m_e^2}d\varepsilon = \frac{1}{\pi^2}\int_{m_e}^{\varepsilon_F}\varepsilon\sqrt{\varepsilon^2 - m_e^2}d\varepsilon = \frac{1}{3\pi^2}\left(\varepsilon_F^2 - m_e^2\right)^{3/2} \tag{8}$$

Where the $T \to 0$ limit was taken.
From chemical equilibrium and the fact that $\varepsilon_F = \mu_\pi \leq \varepsilon_{min} = m_\pi$ it is derived that:

$$n_e = \frac{1}{3\pi^2}\left(\varepsilon_F^2 - m_e^2\right)^{3/2} = \frac{1}{3\pi^2}\left(\mu_\pi^2 - m_e^2\right)^{3/2} \leq \frac{1}{3\pi^2}\left(m_\pi^2 - m_e^2\right)^{3/2} \tag{9}$$

Separation of cases:
1. If $m_\pi < m_e$ then $\varepsilon_F < m_e$ leading to $n_e = 0$, and $n_\pi = \bar{n} - n_e = \bar{n}$. Thus, the case of $T = 0$ yields in a pure pion gas in the ground state as a BEC, note the left scenario in Figure 1.
2. If $m_\pi > m_e$ then n_e is finite, thus the electrons occupy the energy levels between m_e to ε_F. As can be seen from the inequality above, if $m_\pi > \varepsilon_F(\bar{n})$ then $n_e = \bar{n}$ and $n_\pi = 0$ as it is shown in the middle scenario in Figure 1. Else, $m_e < m_\pi < \varepsilon_F(\bar{n})$ leading to occupation of the electrons up to the energy of m_π, meaning $n_e = \frac{1}{3\pi^2}\left(m_\pi^2 - m_e^2\right)^{3/2}$. The other particles would occupy as bosons the ground state as a BEC as noted in the right scenario in Figure 1 and their density would be $n_\pi = \bar{n} - n_e = \bar{n} - \frac{1}{3\pi^2}\left(m_\pi^2 - m_e^2\right)^{3/2}$.