Ex4450: Beta decay $n \to e + p + \bar{\nu}$

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The problem:

In this problem we regard (anti)neutrino as massless spin 1/2 fermions. If we place neutrons in some confined region, some of them β-decay into protons, electrons and antineutrinos via the reaction $n \Leftrightarrow p + e^{-} + \bar{\nu}$. The masses are $\mathsf{m}_n, \mathsf{m}_p, \mathsf{m}_e$. The objective is to estimate the final equilibrium density of n of the neutrons, given that that initially density was n_0.

(a) Find the equations of states for gas of massless spin 1/2 fermions. Namely, express the density of particles and the density of energy as a function of the chemical potential μ, the temperature T, and the speed of light c. Evaluate the integrals for $T = 0$ and for large T. Define what does it mean large T.

(b) Write equations for the neutron density n, given that initially there were only neutrons with density n_0, and the temperature is T. Consider appriximation under the following assumptions:

(b1) The particles are non-relativistic (except the nutrino)

(b2) The particles are hyper relativistic (negligible mass).

(b3) The temperature is zero.

(b4) The temperature is high (Boltzmann approximation).

The solution:

(a) $\varepsilon = \hbar c k$

$\varepsilon^2 = \hbar^2 c^2 k^2 = \frac{(2\pi)^2 \hbar^2 c^2}{L^2}(n_x^2 + n_y^2 + n_z^2)$

$N(\varepsilon) = 2 \times \frac{3\pi}{4}(\frac{L\varepsilon}{2\pi\hbar c})^3 = \frac{V\varepsilon^3}{3\pi\hbar^3 c^3}$

$g(\varepsilon) = \frac{V\varepsilon^2}{\pi\hbar^3 c^3}$

$\frac{E}{V} = \int_0^\infty \frac{\varepsilon^3 d\varepsilon}{\pi\hbar^3 c^3 (1+e^{\beta(\varepsilon-\mu)})} = \frac{T^4}{\pi\hbar^3 c^3}\int_0^\infty \frac{x^3 dx}{1+e^{x-\mu\beta}}$

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When $T = 0$:

$\frac{E}{V} = \int_0^\infty \frac{\varepsilon^3 d\varepsilon}{\pi\hbar^3 c^3 (1+e^{\beta(\varepsilon-\mu)})} \approx \int_0^\mu \frac{\varepsilon^3 d\varepsilon}{\pi\hbar^3 c^3} = \frac{\mu^4}{4\pi\hbar^3 c^3}$

$\frac{N}{V} = \int_0^\infty \frac{\varepsilon^2 d\varepsilon}{\pi\hbar^3 c^3 (1+e^{\beta(\varepsilon-\mu)})} \approx \int_0^\mu \frac{\varepsilon^2 d\varepsilon}{\pi\hbar^3 c^3} = \frac{\mu^3}{3\pi\hbar^3 c^3}$

When $T >> \mu$ ($\beta\mu << 1$):

$\frac{E}{V} = \frac{T^4}{\pi\hbar^3 c^3}\int_0^\infty \frac{x^3 dx}{1+e^{x-\mu\beta}} \approx \frac{T^4}{\pi\hbar^3 c^3}\int_0^\infty \frac{x^3 dx}{1+e^{x}}$

$\frac{N}{V} = \frac{T^3}{\pi\hbar^3 c^3}\int_0^\infty \frac{x^2 dx}{1+e^{x-\mu\beta}} \approx \frac{T^3}{\pi\hbar^3 c^3}\int_0^\infty \frac{x^2 dx}{1+e^{x}}$

$\int_0^\infty \frac{x^n dx}{1+e^x} = \int_0^\infty (\frac{-2x^n}{e^{2x}-1} + \frac{x^n}{e^x-1})dx = (1 - \frac{1}{2^n})\int_0^\infty \frac{x^n dx}{e^x-1}$

$= (1 - \frac{1}{2^n})\Gamma(n+1)\zeta(n+1) = (1 - \frac{1}{2^n})n!\zeta(n+1)$

$\zeta(4) = \frac{\pi^4}{90}$

$\frac{E}{V} = \frac{7\pi^3 T^4}{120\hbar^3 c^3}$

$\frac{N}{V} = \frac{\zeta(3) T^3}{\pi\hbar^3 c^3}$

(b) $\varepsilon^2 = \hbar^2 c^2 k^2 + m_0^2 c^4$

$N(\varepsilon) = 2 \times \frac{4\pi}{3}(\frac{L}{2\pi\hbar c})^3(\varepsilon^2 - m_0^2 c^4)^{\frac{3}{2}} = \frac{V}{3\pi\hbar^3 c^3}(\varepsilon^2 - m_0^2 c^4)^{\frac{3}{2}}$

$g(\varepsilon) = \frac{V\varepsilon}{\pi\hbar^3 c^3}(\varepsilon^2 - m_0^2 c^4)^{\frac{1}{2}}$

$n = \frac{1}{\pi\hbar^3 c^3}\int_{m_n c^2}^{\infty} \frac{\varepsilon(\varepsilon^2 - m_n^2 c^4)^{\frac{1}{2}} d\varepsilon}{1+e^{\beta(\varepsilon-\mu_n)}}$

$n_0 - n = \frac{1}{\pi\hbar^3 c^3}\int_{m_e c^2}^{\infty} \frac{\varepsilon(\varepsilon^2 - m_e^2 c^4)^{\frac{1}{2}} d\varepsilon}{1+e^{\beta(\varepsilon-\mu_{e^-})}}$

$n_0 - n = \frac{1}{\pi\hbar^3 c^3}\int_{m_p c^2}^{\infty} \frac{\varepsilon(\varepsilon^2 - m_p^2 c^4)^{\frac{1}{2}} d\varepsilon}{1+e^{\beta(\varepsilon-\mu_p)}}$

$n_0 - n = \frac{1}{\pi\hbar^3 c^3}\int_{0}^{\infty} \frac{\varepsilon^2 d\varepsilon}{1+e^{\beta(\varepsilon-\mu_\nu)}}$

$\mu_n(n_0 - n) = \mu_p(n) + \mu_{e^-}(n) + \mu_{\bar{\nu}}(n)$

The beta decay reaction is:

$$n \Leftrightarrow e^- + p + \bar{\nu}$$

This means that at equilibrium, the following relation between the chemical potentials of the different gases must hold :

$$\mu_n(n_0 - n) = \mu_p(n) + \mu_{e^-}(n) + \mu_{\bar{\nu}}(n)$$

where n is the number of beta decays.
Since all of the involved particles are spin 1/2 fermions, therefore obey the same particle statistics and dispersion relations (besides the anti-neutrinos in most cases), we only need to evaluate one integral at a time of the form :

$$N = \int_0^\infty f(\epsilon - \mu) g(\epsilon) d\epsilon$$

$$g(\vec{p}) = \frac{V}{\pi^2\hbar^3} p^2$$

$$f(\epsilon - \mu) = \left(e^{\beta(\epsilon-\mu)} + 1\right)^{-1}$$

(b1) Non - relativistic : $E = p^2/2m$

$$g(\epsilon)d\epsilon = \frac{\sqrt{2m^3}\, V}{\pi^2\hbar^3}\sqrt{\epsilon}$$

(b2) Hyper - relativistic : $E = pc$

$$g(\epsilon)d\epsilon = \frac{V}{\pi^2(\hbar c)^3}\epsilon^2$$

(b3) Zero temperature : $\beta \to \infty$

$f(\epsilon - \mu) = \theta(\mu - \epsilon)$

For the non-relativistic fermions :

$A = \frac{\sqrt{2}}{\pi^2\hbar^3}$

$$\frac{N}{V} = n = Am^{\frac{3}{2}} \int_0^{\mu} \sqrt{\epsilon} d\epsilon = \frac{2}{3} A \left(m\mu\right)^{\frac{3}{2}}$$
$$\mu = \frac{1}{m} \left(\frac{3n}{2A}\right)^{2/3}$$

For the hyper-relativistic fermions :

$$B = \frac{1}{\pi^2 (\hbar c)^3}$$
$$\frac{N}{V} = n = B \int_0^{\mu} \epsilon^2 d\epsilon = \frac{B}{3}\, \mu^3$$
$$\mu = \left(\frac{3n}{B}\right)^{\frac{1}{3}}$$

In the general case :

$$E = c\sqrt{p^2 + m^2c^2}$$
$$p = \frac{1}{c}\sqrt{E^2 - m^2c^4}$$
$$\frac{dp}{dE} = \frac{1}{c} \frac{E}{\sqrt{E^2 - m^2c^4}} = \frac{E}{pc^2}$$
$$g(\epsilon)d\epsilon = BV\epsilon\sqrt{\epsilon^2 - m^2c^4}\; d\epsilon$$

$$n = \int_{mc^2}^{\mu + mc^2} B\epsilon\sqrt{\epsilon^2 - m^2c^4}\; d\epsilon = \frac{B}{3} \left(\epsilon^2 - m^2c^4\right)^{3/2} |_{mc^2}^{\mu+mc^2} = \frac{B}{3} \left(\mu(\mu + 2mc^2)\right)^{3/2}$$

Solving for μ :

$$\mu(n) = \sqrt{m^2c^4 + \left(\tfrac{3n}{B}\right)^{2/3}} - mc^2$$

If all the particles are non-relativistic (except the anti-neutrino), then:

$$\left(\frac{1}{m_p} + \frac{1}{m_e}\right) \left(\frac{3n}{2A}\right)^{2/3} + \left(\frac{3n}{B}\right)^{\frac{1}{3}} = \frac{1}{m_n} \left(\frac{3(n_0 - n)}{2A}\right)^{2/3}$$

If all the particles are hyper-relativistic, then:

$$3\left(\frac{3n}{B}\right)^{\frac{1}{3}} = \left(\frac{3(n_0 - n)}{B}\right)^{\frac{1}{3}}$$

Solving for n, we get :

$$n = \frac{n_0}{28}$$

(b4) High temperature :

$$f(\epsilon - \mu) \approx e^{-\beta(\epsilon - \mu)}$$
$$n = B\int_{mc^2}^{\infty} e^{-\beta(\epsilon-\mu)} \epsilon\sqrt{\epsilon^2 - m^2c^4}\; d\epsilon$$

change of variables : $y = \sqrt{\epsilon^2 - m^2c^4}$; $d\epsilon = \frac{y}{\epsilon} dy$

$$n = \frac{B}{3} e^{\beta\mu} \int_0^\infty e^{-\beta\sqrt{y^2+m^2c^4}} y^2 dy$$

non-relativistic limit : $\sqrt{m^2c^4 + y^2} \approx mc^2(1 + \frac{y^2}{2mc^2})$

$$n = \frac{B}{3} e^{\beta(\mu - mc^2)} \int_0^\infty e^{-\dfrac{\beta}{2mc^2} y^2} y^2 dy$$

$$n = \frac{B}{3} \left(\frac{2mc^2}{\beta}\right)^{3/2} e^{\beta(\mu - mc^2)} \int_0^\infty e^{-z^2} z^2 dz = \frac{B\sqrt{\pi}}{12} \left(\frac{2mc^2}{\beta}\right)^{3/2} e^{\beta(\mu - mc^2)}$$

$$\mu(n) = mc^2 + k_B T \left[\tfrac{3}{2} ln\left(\tfrac{\pi\hbar^2}{mk_B T}\right) + ln\,(3n)\right]$$

Hyper-relativistic limit : $\sqrt{m^2c^4 + y^2} \approx |y|$

$$n = \frac{B}{3} e^{\beta\mu} \int_0^\infty e^{-\beta y} y^2 dy = \frac{2B}{3\beta^3} e^{\beta\mu}$$

$$\mu(n) = k_B T ln\left(\frac{3n}{2B(k_B T)^3}\right)$$

The chemical potential equation is therefore:

$$3k_B T ln\left(\frac{3n}{2B(k_B T)^3}\right) = k_B T ln\left(\frac{3(n_0 - n)}{2B(k_B T)^3}\right)$$

$$n^3 = \left(\frac{2B}{3}\right)^2 \frac{n_0 - n}{\beta^6}$$