Ex4442: Chemical equilibrium for gamma == e and gamma == pi

Submitted by: Assaf Lavi, Ofir Arad

The problem:

The reaction $\gamma + \gamma \leftrightarrow e^+ + e^-$ occurs inside a star, where γ is a photon and $e^\pm$ are the positron and electron (mass m). In general $e^\pm$ are relativistic. Assume overall charge neutrality, and that the system is in equilibrium at temperature T.

(1) Find an expression for the densities of $e^\pm$
(2) Find these densities in the limit $T \ll mc^2$
(3) Solve the same problem for the reaction $\gamma + \gamma \leftrightarrow \pi^+ + \pi^-$, where π are bosons with mass M
(4) Can the π bosons become Bose-condensed if the temperature is sufficiently lowered? Explain the result physically.

The solution:

(1) From equilibrium we deduce that $2\mu_\gamma = \mu_{e^+} + \mu_{e^-}$. Since photons have zero chemical potential we define $\mu \equiv \mu_{e^-}$ and as a result $\mu = -\mu_{e^+}$. The density of fermions can be calculated from the occupation function $f(\epsilon - \mu)$. For a relativistic particle, the energy is given by $\epsilon = mc^2\sqrt{1 + (p/mc)^2}$, with m the electron mass, and for electrons/positrons, the spin gives a factor of 2 in the occupation, so we get

$$n_{e^\pm} = \frac{2}{V}\sum_p \frac{1}{e^{\beta mc^2\sqrt{1+(p/mc)^2} \pm \beta\mu} + 1} \ . \tag{1}$$

From charge neutrality we know that these densities are equal, $n_{e^-} = n_{e^+}$. Noting that every term in the series in Eq. (1) is monotonic in μ, charge neutrality demands $\mu = 0$.
We can also write the densities as an integral, using $\frac{(2\pi)^3}{V}$ as the spacing in momentum space,

$$n_e = \frac{2}{(2\pi)^3}\int_0^\infty \frac{1}{e^{\beta mc^2\sqrt{1+(p/mc)^2}} + 1} d^3p \ . \tag{2}$$

(2) In the limit $\beta mc^2 \gg 1$, the main contributions in the integral in Eq. (2) come from small p. Thus

we can approximate the particles energies as non-relativistic and use the Boltzmann approximation to yield

$$\epsilon \approx mc^2 + \frac{p^2}{2m} \ , \tag{3}$$

$$n_e \simeq \frac{2}{(2\pi)^3}\int_0^\infty \frac{1}{e^{\beta(mc^2 + p^2/2m)} + 1} d^3p \simeq \frac{2e^{-\beta mc^2}}{(2\pi)^3}\int_0^\infty e^{-\frac{\beta p^2}{2m}} d^3p = \frac{2e^{-\beta mc^2}}{\lambda_T^3} \ ,$$

where $\lambda_T \equiv \left(\frac{m}{2\pi\beta}\right)^{-1/2}$. This result is similar to what we would expect from a classical ideal gas.

(3) Now we have $\pi^\pm$ bosons, with mass M, so the occupation function changes to Bose-Einstein, and the densities become

$$n_{\pi^\pm} = \frac{1}{V}\sum_p \frac{1}{e^{\beta Mc^2\sqrt{1+(p/Mc)^2} \pm \beta\mu} - 1} \ , \tag{4}$$

where $\mu \equiv \mu_{\pi^-} = -\mu_{\pi^+}$. For bosons, the chemical potential must be smaller than the ground state energy, which in our case demands $-Mc^2 < \mu < Mc^2$. In that region, similarly to the fermionic case, each term in the series in Eq. (4) is monotonic in μ, so charge neutrality assures us that $\mu = 0$ here too.
Writing the densities as integrals, we have

$$n_\pi = \frac{1}{(2\pi)^3} \int \frac{1}{e^{\beta Mc^2 \sqrt{1+(p/Mc)^2}} - 1} d^3p \ . \tag{5}$$

In the $T \ll Mc^2$ limit, the same logic applies to justify the Boltzmann approximation, and we have

$$n_\pi \simeq \frac{e^{-\beta Mc^2}}{(2\pi)^3} \int_0^\infty e^{-\frac{\beta p^2}{2M}} d^3p = \frac{e^{-\beta Mc^2}}{\lambda_T^3} \ . \tag{6}$$

(4) Since we showed that charge neutrality demands $\mu = 0$, no condensation is possible. In more detail - to have condensation, the chemical potential must tend to the ground state energy, which in our case would mean $\mu \to \pm Mc^2$, but this is impossible and would mean a net charge in the system.
To understand what happens physically, we examine Eq. (6),

$$n_\pi \simeq \frac{e^{-\beta Mc^2}}{\lambda_T^3} \propto \frac{e^{-Mc^2/T}}{T^{3/2}} \tag{7}$$

so when we lower the temperature, the densities become smaller and smaller, i.e. there are less particles instead of the particles being in condensation.