E3745: Fermions in a gravitational field

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The problem:

Consider fermions of mass m and spin $1/2$ in a gravitational field with constant acceleration g and at uniform temperature T.

(1) Assume first that the fermions behave as classical particles and find their density $n(h)$ as function of the height $h \geq 0$ and the density $n(0) = n_0$.

(2) Near the bottom $h = 0$ the fermions are degenerate, i.e. their Fermi energy $\epsilon_F^0 \equiv \epsilon_F(h = 0)$ is $\epsilon_F^0 \geqq T$. Assume $T = 0$, Find the local Fermi momentum $p_F(h)$ and $n(h)$ given n_0.

(3) Consider now $T \neq 0$ (but still $\epsilon_F^0 \geqq T$) and estimate the height h_c such that for $h \gg h_c$ the fermions are non-degenerate. Find $n(h)$ for $h \gg h_c$ in terms of n_0.

The solution:

(1) The Partition function for a layer of a uniform density positioned at a height $[h, h+\delta h$ is just the regular classical gas partition function multiplied by an exponential factor containing the dependence in the height of the element

$$Z_1(\beta, h, \delta h) = \int_0^L \int_0^L \int_h^{h+\delta h} \int_{-\infty}^{+\infty} 2(e^{-(\frac{\beta p^2}{2m}+\beta mgz)})\frac{dxdydzd^3p}{(\beta mg)(2\pi)^3} = \frac{2L^2}{(2\pi\lambda_T)^3}(e^{-\beta mgh} - e^{-\beta mg(h+\delta h)}) \quad (1)$$

$$= \frac{2L^2}{(\beta mg)(2\pi\lambda_T)^3}e^{-\beta mgh}(1 - e^{-\beta mg\delta h})$$

and since δh is infinitesimally small

$$Z_1(\beta, h, \delta h) = \frac{2L^2}{(2\pi\lambda_T)^3}e^{-\beta mgh}(\delta h) = \frac{2L^2\delta h}{(2\pi\lambda_T)^3}e^{-\beta mgh} \quad (2)$$

We could factorize each particle in order to get

$$Z_N(\beta, h, \delta h) = \frac{1}{N(h)!}(Z_1)^N \quad (3)$$

Deriving the Free energy

$$F = TN(h)(-1 + ln\frac{(2\pi\lambda_T)^3 N(h)}{2L^2\delta h} + \beta mgh) \quad (4)$$

and the chemical potential

$$\mu(\beta, h) = T(ln\frac{(2\pi\lambda_T)^3 n(h)}{2} + \beta mgh) \quad (5)$$

By demand for chemical equilibrium we get $\mu(h) = \mu(0)$ hence

$$T(ln\frac{(2\pi\lambda_T)^3 n(h)}{2}) + mgh = T(ln\frac{(2\pi\lambda_T)^3 n(0)}{2}) \quad (6)$$

$$n(h) = n(0)e^{-\beta mgh} \tag{7}$$

(2) While $T = 0$ since $\lambda_T \sim \frac{1}{\sqrt{mT}} \to \infty$ we get overlap between the different fermionic wave functions a.k.a degeneration.
Counting the Number of states in the Fermi sphere

$$N = 2 \times \frac{1}{8} \times \frac{4}{3}\pi n_f^3 \tag{8}$$

from $n_f = (\frac{3N}{\pi})^{1/3}$ we get the Fermi energy

$$\epsilon_F = \frac{\pi^2}{2mL^2}n_f^2 = \frac{1}{2m}(\frac{3\pi^2 N}{V})^{2/3} \tag{9}$$

The chemical potential just equals ϵ_F^0 so we have

$$\epsilon_F^0 - mgh = \frac{p_F(h)^2}{2m} = \frac{1}{2m}(3\pi^2 n(h))^{2/3} \tag{10}$$

from substituting $h = 0$ we get $\epsilon_F^0 = \frac{1}{2m}(3\pi^2 n(0))^{2/3}$ and

$$\frac{1}{2m}(3\pi^2 n(h))^{2/3} = \frac{1}{2m}(3\pi^2 n(0))^{2/3} - mgh \tag{11}$$

$$3\pi^2 n(h) = (3\pi^2 n(0))^{2/3} - 2m^2 gh)^{3/2} = (2m)^{3/2}(\epsilon_F^0 - mgh)^{3/2} \tag{12}$$

(3) The degeneration would be maintained as long as $\epsilon_F \gg T$, remembering

$$\epsilon_F^0 \equiv \epsilon_F(h = 0) \tag{13}$$

the critical height would provide

$$\epsilon_F = \epsilon_F^0 - mgh = T \tag{14}$$

but since $\epsilon_F^0 \gg T$ the critical height would be

$$\epsilon_F^0 \approx mgh_c \tag{15}$$

$$h_c = \frac{\epsilon_F^0}{mg} \tag{16}$$

we can get μ from the condition on $h = 0$ (since we are still at a very low temperature)

$$\mu = \epsilon_F^0 = \frac{1}{2m}(3\pi^2 n(0))^{2/3} \tag{17}$$

substituting into the expression for the chemical potential at the classical region $h > h_c$

$$\frac{1}{2m}(3\pi^2 n(0))^{2/3} = T(ln\frac{(2\pi\lambda_T)^3 n(h)}{2} + \beta mgh) \tag{18}$$

$$n(h) = \frac{2}{(2\pi\lambda_T)^3}e^{\frac{\beta}{2m}(3\pi^2 n(0))^{2/3} - \beta mgh} \tag{19}$$