Ex3711: Fermions in gravitation field of a star

Submitted by: Barak Azoulai

The problem:

Consider a neutron star as non-relativistic gas of non-interacting neutrons of mass m in a spherical symmetric equilibrium configuration. The neutrons are held together by a gravitational potential $-mMG/r$ of a heavy object of mass M and radius r_0 at the center of the star (G is the gravity constant and r is the distance from the center).

(a) Give an expression for $n(r)$ at $T > 0$ using Li functions.

(b) Consider the neutrons as fermions at $T = 0$ and find $n(r)$, for a given $n(r_0)$.

(c) Calculate it explicitly in the Boltzmann approximation.

(d) Repeat items (b) and (c) for a general potential $-A/r^\alpha$.

(e) For the case of T=0, what is the upper bound on $n(r_0)$ and on the total number N of neutrons if the chemical potential is increased towards zero. Distinguish, $a > 2$ from $a < 2$.

The solution:
This problem is about neutrons (which are fermions) in a gravitational field, one can define a small volume dV, take advantage of the fact that the neutrons are in an equilibrium configuration and demand that the chemical potential μ is the same for each volume dV.

(a) The density of particles is defined such that the number of particles in a volume dV is $N(r) = n(r)dV$, which is calculated by $N(r) = \int g(\epsilon)f(\epsilon-\mu)d\epsilon$. The energies of the one particle Hamiltonian at a specific radius r in a box of volume dV are:

$$\epsilon(r) = \frac{p^2}{2m} - U(r). \tag{1}$$

With $U(r) = \frac{mMG}{r}$. The density of states in a volume dV is:

$$g(\epsilon) = 2 \cdot 4\pi m^{3/2} \left[2\left(\epsilon + U(r)\right)\right]^{1/2} \cdot \frac{dV}{(2\pi)^3} = \frac{(2m)^{3/2}}{2\pi^2} \left(\epsilon + U(r)\right)^{1/2} dV. \tag{2}$$

For fermions the particle distribution $f(\epsilon-\mu) = \left(e^{\beta(\epsilon-\mu)} + 1\right)^{-1}$, and the general solution for the density of the particles is given by:

$$n(r) = \frac{(2m)^{3/2}}{2\pi^2} \int\limits_{-U(r)}^{\infty} \frac{\left(\epsilon + U(r)\right)^{1/2}}{e^{\beta(\epsilon-\mu)} + 1} d\epsilon = \frac{(2mT)^{3/2}}{2\pi^2} \int\limits_0^{\infty} \frac{x^{\alpha-1}dx}{e^{x-u} + 1} = -\frac{(2mT)^{3/2}}{2\pi^2} \Gamma(3/2)\mathrm{Li}_{3/2}(-e^u) \tag{3}$$

With $x = \beta\left(\epsilon + U(r)\right)$, $\alpha = 3/2$, $u = \beta\left(\mu + U(r)\right)$ and using $\int\limits_0^{\infty} \frac{x^{\alpha-1}dx}{e^{x-u}+1} = -\Gamma(\alpha)\mathrm{Li}_\alpha(-e^u)$

(b) At the limit of $T = 0$, one can take Eq. (3) and use the expansion $\Gamma(\alpha)\mathrm{Li}_\alpha(-e^u) \approx -\frac{1}{\alpha}u^\alpha$, or alternatively calculate $n(r)$ explicitly and noting that $f(\epsilon - \mu)$ becomes a step function $\Theta(\mu - \epsilon)$.

$$n(r) = \frac{(2m)^{3/2}}{2\pi^2} \int\limits_{-U(r)}^{\infty} \Theta(\mu - \epsilon)\left(\epsilon + U(r)\right)^{1/2} d\epsilon = \frac{(2m)^{3/2}}{3\pi^2} \left(\mu + U(r)\right)^{3/2} \tag{4}$$

To obtain μ we set $r = r_0$ in Eq. (4) and express it with $n(r_0)$, one obtains:

$$\mu = \frac{\left(3\pi^2 n(r_0)\right)^{2/3}}{2m} - U(r_0) \tag{5}$$

For $T = 0$ one gets the density of particles as:

$$n(r) = \frac{(2m)^{3/2}}{3\pi^2}\left[\frac{\left(3\pi^2 n(r_0)\right)^{2/3}}{2m} + U(r) - U(r_0)\right]^{3/2} \tag{6}$$

(c) At the Boltzmann approximation, one can take the limit of Li function $\mathrm{Li}(-e^u) \approx -e^u$ and get:

$$n(r) = -\frac{(2mT)^{3/2}}{2\pi^2}\Gamma(3/2)\mathrm{Li}_{3/2}(-e^u) \tag{7}$$

$$\approx \frac{(2mT)^{3/2}}{2\pi^2}\frac{\sqrt{\pi}}{2}\exp\left[\beta\left(\mu + U(r)\right)\right] = \frac{2}{\lambda_T^3}\exp\left[\beta\left(\mu + U(r)\right)\right] \tag{8}$$

Where we used $(\lambda_T)^2 = 2\pi/mT$, and $\Gamma(3/2) = \sqrt{\pi}/2$. The same result can be obtained by performing the integral for $n(r)$ explicitly and noting that $f(\epsilon - \mu) = \left(e^{\beta(\epsilon-\mu)} + 1\right)^{-1} \approx e^{-\beta(\epsilon-\mu)}$. Expressing μ using $n(r_0)$ will give us:

$$\mu = T\ln\left(\frac{1}{2}n(r_0)\lambda_T^3\right) - U(r_0) \tag{9}$$

At the Boltzmann approximation one gets the density of particles as:

$$n(r) = n(r_0)\exp\left[\beta\left(U(r) - U(r_0)\right)\right] \tag{10}$$

Noting that this is the expected result because in canonical equlibrium $p(r) \propto e^{-\beta(\epsilon)}$.

(d) Repeating items (b) and (c) will give the same results but changing r to r^α, so we get the same Eq. (6) & (10) but with

$$U(r) = \frac{A}{r^\alpha} \tag{11}$$

(e) Increasing μ toward zero is done by increasing $n(r_0)$ as one can see from Eq. (5). At the limit of $\mu \to 0$ one obtains:

$$n(r)_{T=0} \underset{\mu\to 0}{=} \frac{(2mA)^{3/2}}{3\pi^2} r^{-3\alpha/2} \tag{12}$$

The total number of neutrons (for $\alpha \neq 2$):

$$N_{\text{total}} = \int_{r_0}^{\infty} n(r)4\pi r^2 dr = \frac{2^{7/2}(mA)^{3/2}}{3\pi}\int_{r_0}^{\infty} r^{2-3\alpha/2}dr \tag{13}$$

$$= \frac{2^{7/2}(mA)^{3/2}}{9\pi}\left(1 - \frac{\alpha}{2}\right)^{-1} r^{3(1-\alpha/2)}\Big|_{r_0}^{\infty} \tag{14}$$

The integral diverges for $\alpha \leq 2$ and converges for $\alpha > 2$ to:

$$N_{\text{total}} = \frac{2^{7/2}(mA)^{3/2}}{9\pi}\left(1 - \frac{\alpha}{2}\right)^{-1} r_0^{3(1-\alpha/2)} \tag{15}$$

For $\mu = 0$ the upper bound for the density $n(r_0)$ is given by

$$n(r_0)_{T=0,\mu=0} = \left(\frac{2mA}{r_0^\alpha}\right)^{3/2}\frac{1}{3\pi^2} \tag{16}$$