Ex3710: Fermions in gravitational field of a star

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The problem:

Consider an artificial model of neutron star where the gas of N neutrons is held together by a gravitational potential $U = -\frac{mMG}{r}$ generated by the solid core of the star. (M and r_0 are the mass and radius of the core, G - gravity constant, m - mass of the neutrons and r - distance from the center).

(1) Assume the neutron gas as Fermi gas. Find the density $n(r)$ at $r > r_0$ for $T = 0$ and the fermi energy $\epsilon_f(N)$.

(2) Find the flux of particles that escapes the gravitational field of the core for low temperature, ($\mu \approx \epsilon_f$).

(3) Write a differential equation for $N(t)$ as a function of time.

(4) Find the the condition in which Boltzmann approximation is valid for all the particles in the system.

The solution:

(1) The number of states

$$N = 2\int_r\int_p \frac{d^3r d^3p}{(2\pi)^3} \quad \Rightarrow \quad n(r) = 2\int_p \frac{d^3p}{(2\pi)^3}$$

After integrating over the angular degrees of freedom we get-

$$n(r) = 2\int_0^{p_f} \frac{4\pi p^2}{(2\pi)^3}dp = 2\frac{4\pi}{3(2\pi)^3}{p_f}^3$$

Where p_f is the fermi momentum.

After substituting $\epsilon = \frac{p^2}{2m} + U \quad \Rightarrow \quad p_f = [2m(\epsilon_f - U)]^{\frac{1}{2}}$ we get

$$n(r) = \frac{(2m)^{\frac{3}{2}}}{3\pi^2}(\epsilon_f + \frac{mMG}{r})^{\frac{3}{2}}$$

The fermi energy can be found from calculating the total number of particles N

$$N = \int_r n(r)d^3r = 4\pi\int_{r_0}^{r_m} n(r)r^2dr = \frac{4}{3\pi}(2m)^{\frac{3}{2}}\int_{r_0}^{r_m}(\epsilon_f + \frac{mMG}{r})^{\frac{3}{2}}r^2dr \quad \Rightarrow \quad \epsilon_f(N)$$

Where r_m is the maximal radius of which there are neutrons, and can be found from the condition $n(r) > 0$

$$(\epsilon_f + \frac{mMG}{r})^{\frac{3}{2}} > 0 \quad \Rightarrow \quad \epsilon_f > -\frac{GMm}{r} \quad \Rightarrow \quad r_m = \frac{GMm}{|\epsilon_f|}$$

(2) For $T > 0$ there is emission of particles (see appendix). In case of low temperature ($\mu \approx \epsilon_f$) the particles that escapes obey Boltzmann statistics, it can be seen from the fugacity

$$z = e^{\beta(\mu+U)} = e^{-\beta(|\epsilon_f| + \frac{GMm}{r})} << 1$$

According to the detailed balance principle the number of particles emitted from the star is the same as the number of particles incident to the star

$$J_e = J_i = J$$

And therefore we can avoid the details of the potential and calculate the incoming free particles that are not influenced by it.

The number of particles at speed $v = (\frac{2\epsilon_k}{m})^{\frac{1}{2}}$ out side the potential U

$$N(v) = V(\frac{m}{2\pi})^3 e^{-\beta(\frac{1}{2}mv^2 - \mu)} 4\pi v^2 dv$$

Which is Boltzmann velocity distribution. Accordingly the incident flux

$$J = \frac{1}{4}\int \frac{N(v)}{V} v dv = \frac{m^3}{(2\pi)^2} e^{\beta\mu} \int_0^\infty e^{-\beta\frac{1}{2}mv^2} v^3 dv = \frac{m}{2(\pi)^{\frac{3}{2}}} T^2 e^{\mu/T} = \frac{m}{2(\pi)^{\frac{3}{2}}} T^2 e^{-|\epsilon_f(N)|/T}$$

We got Richardson law, Where the fermi energy (ϵ_f) is the work function W.

The total flow of emitted newtrons is the flux summed over the emitting surface $I = J 4\pi r_m^2$.

(3)

$$I = \frac{dN(t)}{dt} = \frac{2r_m^2}{\pi^{1/2}} m T^2 e^{-|\epsilon_f(N(t))|/T}$$

We can see from this equation that N doesn't stay constant, therefore the system is out of equilibrium with the environment and Richardson law will be a good approximation only for a short time after the change of temperature from $T = 0$ to $T > 0$.

(4) The condition for Boltzmann approximation

$$z = e^{\beta(\mu-U)} = e^{\beta(-|\mu| + \frac{GMm}{r})} \ll 1 \quad \Rightarrow \quad T \ll T_f - \frac{GMm}{r} \quad \Rightarrow \quad T \ll T_f - \frac{GMm}{r_0}$$

The last equality says that if Boltzmann approximation is valid at the surface of the core (r_0), then it will be valid at any distance ($r > r_0$) as well.

Appendix - The density of particles at radius r and $T > 0$.

The total number of particles is given by the integral over the fermi distribution, with factor of 2 for spin $\frac{1}{2}$ degeneracy.

$$N = 2\int_r\int_p \frac{d^3rd^3p}{(2\pi)^3}\frac{1}{e^{\beta(\frac{p^2}{2m}+U-\mu)}+1} = \frac{2}{(2\pi)^3}\int_{r_0}^{\infty} d^3r\int_{-\infty}^{\infty} d^3p\frac{1}{\frac{1}{z}e^{\beta(\frac{p^2}{2m})}+1}$$

Where $z = e^{\beta(\mu-U)}$ is the fugacity and $U = -\frac{mMG}{r}$.

The number of particles in a volume element at radius r

$$n(r) = \frac{2}{(2\pi)^3}\int_{-\infty}^{\infty} d^3p\frac{1}{\frac{1}{z}e^{\beta(\frac{p^2}{2m})}+1}$$

After substituting $p = (2m\epsilon_k)^{\frac{1}{2}} \quad \Rightarrow \quad p^2dp = \frac{1}{2}(2m)^{\frac{3}{2}}(\epsilon_k)^{\frac{1}{2}}$, Where ϵ_k is the kinetic energy, and integrating over the angular degrees of freedom we get

$$n(r) = \frac{(2m)^{\frac{3}{2}}}{2\pi^2}\int_0^{\infty} d\epsilon_k\frac{\epsilon_k^{\frac{1}{2}}}{\frac{1}{z}e^{\beta\epsilon_k}+1}$$

Switching to dimensionless paramater $x = \beta\epsilon \quad \Rightarrow \quad \epsilon_k^{\frac{1}{2}}d\epsilon_k = \frac{1}{\beta^{\frac{3}{2}}}x^{\frac{1}{2}}dx$

$$n(r) = \frac{1}{2\pi^2}(\frac{2m}{\beta})^{\frac{3}{2}}\int_0^{\infty} dx\frac{x^{\frac{1}{2}}}{\frac{1}{z}e^x+1} = \frac{1}{2\pi^2}(\frac{2m}{\beta})^{\frac{3}{2}}F_{\frac{3}{2}}(z)$$

Observing the Last expression we see that $n(r \to \infty) > 0$ which says there is emission of particles for $T > 0$. For $T = 0$ we get same result as section [1] i.e. no emmition.