E3570: Fermi gas in 2D+3D connected boxes with gravitation

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The problem:

Consider a mesoscopic box that has dimensions $L \times L \times \ell$, such that $\ell \ll L$. In the box there are N spin 1/2 electrons. The mass of an electron is m. In items (a-d) assume that the temperature is $T = 0$. In items (1-4) the box is attached to a tank that has dimensions $L \times L \times L$, that is placed at height D relative to the box, and you have to take into account the gravitational field g. Express your answers using m, L, ℓ, g, T.

(1) Describe the single particle density of states. Specify the energy range over which it is the same as for a two dimensional box.
(2) Find the fermi energy E_F assuming that it is in the range defined above. What is the maximum number N_{max} of electrons that can be accommodated without violating the $2D$ description?
(3) Assuming $N < N_{max}$ find the pressure P on the side walls of the box, and the force F on the horizontal walls.
(4) Assume $N = N_{max}$. What is the minimum height D_{min} to place the tank such that all of the electrons stay in the box?
(5) Assume $N = N_{max}$ and $D > D_{min}$. The temperature of the system is raised a little bit. As a result some of the particles that were in the $2D$ box are transferred to the tank. Estimate their number N'. You are allowed to use any reasonable approximation.

The solution:

(1) $3D$ density of states for spin $\frac{1}{2}$ fermions is:

$$g_{3D}(\varepsilon) = 2L^2 l \cdot \frac{(2m)^{\frac{3}{2}}}{(2\pi)^2} \varepsilon^{\frac{1}{2}} \tag{1}$$

In $2D$ however, density of states is independent of energy:

$$g_{2D}(\varepsilon) = \frac{mL^2}{\pi} \tag{2}$$

The condition on the energy for $2D$ approximation of $3D$ box is that it is lower then the first excited state of the "short" axis:

$$\frac{\pi^2}{mL^2} + \frac{\pi^2}{2ml^2} \leq \varepsilon \leq \frac{4\pi^2}{2ml^2} + \frac{\pi^2}{mL^2} \tag{3}$$

(2) The maximal number of particles in $2D$ box is given by:

$$N = \int_0^\infty g(\varepsilon) f(\varepsilon - \mu) d\varepsilon = \int_0^\infty \frac{mL^2}{\pi} \cdot \frac{1}{\frac{1}{z} e^{\beta\varepsilon} + 1} d\varepsilon = \frac{L^2 m}{\pi} kT \ln(1 + e^{\beta\mu}) \tag{4}$$

For $T \to 0$:

$$\lim_{T \to 0} kT \ln(1 + e^{\beta\mu}) = \mu \tag{5}$$

So the number of particles at temprature $T = 0$ is:

$$N = \frac{L^2 m\mu}{\pi} \tag{6}$$

Also at $T = 0$ the chemical potential is defined as:

$$\varepsilon_F = \mu \tag{7}$$

So:

$$\varepsilon_F = \frac{\pi N}{mL^2} \tag{8}$$

and the maximal number of particles under this approximation is:

$$N_{max} = \frac{L^2 m}{\pi}\left(\frac{4\pi^2}{2ml^2} + \frac{\pi^2}{mL^2}\right) \approx 2\pi\frac{L^2}{l^2} \tag{9}$$

(3) First we notice that at $T = 0$, $F = E$.
The single particle energy function:

$$\varepsilon_{k_x,k_x,n_z} = k_x^2 + k_y^2 + \frac{1}{2m}\left(\frac{\pi}{l}n_z\right)^2 \tag{10}$$

We notice that the system is at the ground state for it's "short" axis (it is only dependent on l) with $n_z = 1$. The force on the horizontal walls is:

$$F = -\frac{\partial E}{\partial l} = -N\frac{\partial(\frac{\pi^2}{2ml^2})}{\partial l} = \frac{N\pi^2}{ml^3} \tag{11}$$

The energy is given by:

$$E = \int_0^\infty g_{2D}(\varepsilon)\cdot\varepsilon\cdot f(\varepsilon-\mu)d\varepsilon = \int_0^\infty \frac{mL^2}{\pi}\cdot\frac{\varepsilon}{\frac{1}{z}e^{\beta\varepsilon}+1}d\varepsilon \tag{12}$$

By taking $T = 0$ and $\mu = \varepsilon_F$ we get:

$$E = \frac{\pi}{2m}\frac{N^2}{L^2} \tag{13}$$

and the force on the side walls is:

$$F = -\frac{\partial E}{\partial L} = -\frac{\partial(\frac{\pi}{2m}\frac{N^2}{L^2})}{\partial L} = \frac{\pi N^2}{mL^3} \tag{14}$$

The pressure is given by:

$$P = \frac{F}{A} = \frac{\pi N^2}{AmL^3} = \frac{\pi N^2}{mL^4 l} \tag{15}$$

(4) The condition for no transition between the box and the tank is that the chemical potential of the unoccupied tank is higher then the chemical potential of the box:

$$\mu_{box} = \varepsilon_F = \frac{\pi N_{max}}{mL^2} \tag{16}$$

$$\mu_{tank}(N' = 0) = mgd \tag{17}$$

$$\mu_{box} < \mu_{tank}(N' = 0) \tag{18}$$

$$d_{min} > \frac{\pi N_{max}}{m^2 L^2 g} \tag{19}$$

(5) Now we assume $T > 0$, $d > d_{min}$. The number of particles in the box is given by:

$$N = \frac{L^2 m}{\pi} kT \ln(1 + e^{\beta\mu}) \tag{20}$$

$$\mu = kT \ln(e^{\frac{\pi N}{L^2 mkT}} - 1) = kT \ln(e^{\frac{\varepsilon_f}{kT}} - 1) \tag{21}$$

$$\mu = kT \ln\left(e^{\frac{\varepsilon_f}{kT}}\left(1 - e^{-\frac{\varepsilon_f}{kT}}\right)\right) = kT\left(\frac{\varepsilon_f}{kT}\right) + kT \ln\left(1 - e^{-\frac{\varepsilon_f}{kT}}\right) \tag{22}$$

By using Taylor expansion near $T = 0$, (Note:$e^{-\frac{\varepsilon_f}{kT}} \to 0$; taylor expansion for $\ln(1-x) \sim \ln\left(\frac{1}{1+x}\right) \sim -\ln(1+x) \sim -x$):

$$\mu_{box} = \varepsilon_f - kTe^{-\frac{\varepsilon_f}{kT}} \tag{23}$$

We assume low occupation in the the tank so Boltzmann approximation is valid (and also $N - N' \approx N$)

$$\mu_{tank} = mgd + kT \ln(\frac{N'}{V}\lambda_T^3) \tag{24}$$

We equate the two potential:

$$mgd + kT \ln(\frac{N'}{V}\lambda_T^3) = \varepsilon_F - kTe^{-\frac{\varepsilon_f}{kT}} \tag{25}$$

In order to get an analytical solution for N' we must neglect the exponential term in the chemical potential of the box:

$$N' = \frac{V}{\lambda_T^3} e^{\frac{\varepsilon_F - mgd}{kT}} \tag{26}$$

One can verify that for $T \to 0$, $d = d_{min}$ we get $N' = 0$.