E3570: Fermi gas in 2D+3D connected boxes with gravitation

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The problem:

Given a 'mezoscopic' box with $L \times L \times \ell$. While L is a macroscopic size while ℓ is a mezoscopic $\ell << L$. We insert N fermions with spin $\frac{1}{2}$ and mass m in to a box. in sections a-d assume the temperature is $T = 0$. (mezoscopic size is very small in relation to a large macroscopic size, in relation to microscopic size).

(a) Describe the uniparticle states density and note what is the energy range which the uniparticle states density in it are like a particle in a two dimensional box.

(b) Find the fermi level ε_F in condition it's possible to relate to the box as a two dimensional one. What is the maximum number of N_{max} fermions it's possible to insert in to the box in this condition.

(c) For $N < N_{max}$, as above, find the pressure P on the sided walls of the box, and the force F on the horizontal walls. The box, as above, is attached to a tank with the dimensions $L \times L \times L$ which is D higher from the box. consider gravitation. (assume a very strong gravitation field g, so the question will be reasonable from the order of magnitude).

(d) In the conditions of section b', i.e. $N = N_{max}$, What is the minimum hight D_{min} to place the tank so all of the fermions will stay in the box?

(e) The temperature of the system was raised a little bit. Assume temperature T and also $N = N_{max}, D = D_{min}$. as a result, some of the particles that were in the mezoscopic box, transferred to the tank. Estimate the number N' of these particles. For that, assume that the chemical potential of the system is almost not changing as a result of raising the temperature. Express your answers using $T, \mathsf{g}, \mathsf{m}, L, \ell$ only.

given:

$$\int_0^\infty \frac{x^{\frac{1}{2}}}{e^x + 1} dx = \Gamma\left(\frac{3}{2}\right)\left(1 - \frac{1}{\sqrt{2}}\right)\zeta\left(\frac{3}{2}\right) = 0.678$$

The solution:

(a) Begin with the energy rang, we know that $\varepsilon = \frac{p^2}{2m}$ and the momentum $p_x = \frac{2\pi}{L_x} n_x, p_y = \frac{2\pi}{L_y} n_y, p_z = \frac{2\pi}{L_z} n_z$ so we got that:

$$\varepsilon = \frac{1}{2m}(\frac{2\pi}{L})^2(n_x^2 + n_y^2) + \frac{1}{2m}(\frac{2\pi}{l})^2 n_z^2 \tag{1}$$

after some subsititution we got the energy rang

$$\varepsilon_{range} < \frac{3}{2m}(\frac{2\pi}{l})^2 \tag{2}$$

where $n_z^2 = 2^2 - 1^2 = 3$, and the energy in $2D$

$$\varepsilon_{2D} = \frac{1}{2m}(\frac{2\pi}{L})^2 \tag{3}$$

so now we can define the energy total:

$$\varepsilon = \varepsilon_{2D} + \varepsilon_{range} = \frac{1}{2m}(\frac{2\pi}{L})^2 + \frac{3}{2m}(\frac{2\pi}{l})^2 = \frac{k_L^2 \hbar^2}{2m} + \frac{3k_l^2 \hbar^2}{2m} \tag{4}$$

Actually , being l is very small we can neglect the momentum $k_l = 0$ so $\varepsilon = \frac{k_L^2 \hbar^2}{2m}$.
We know The uniparticle states density is $D(\varepsilon) = \frac{dN}{d\varepsilon}$ where N is the number of the states

$$dN = \frac{2\pi L^2}{(2\pi)^2} k dk \tag{5}$$

$$D(\varepsilon) = \frac{kL^2}{(2\pi)} \frac{dk}{d\varepsilon} \tag{6}$$

we know that $k = \sqrt{\frac{2m\varepsilon}{\hbar}}$ when the DOS is:

$$D(\varepsilon) = \frac{L^2}{(2\pi)} k \frac{dk}{d\varepsilon} = \frac{L^2}{(2\pi)} \frac{\sqrt{2m}}{\hbar} \frac{\sqrt{2m}}{\hbar} \frac{1}{2} \varepsilon^{-\frac{1}{2}} \varepsilon^{\frac{1}{2}} = \frac{L^2 m}{\pi \hbar^2} \tag{7}$$

(b) Being $T = 0$ so:

$$N_{elc.} = \int_0^{\varepsilon_F} \frac{L^2 m}{\pi \hbar^2} d\varepsilon \tag{8}$$

in the end we got Fermi energy:

$$\varepsilon_F = \frac{\pi \hbar^2}{mL^2} N_{elc.} \tag{9}$$

to calculat the N_{max} we use the fact that the maximum of the particals is when $\varepsilon_F = \varepsilon_{gap}$ so

$$N_{max} = \frac{3L^2}{2\pi \hbar^2}(\frac{2\pi}{l})^2 \tag{10}$$

(c) We can use $P = \frac{TK_B}{V}\ln(\mathcal{Z})$, Where the partition function is:

$$\mathcal{Z} = (1 + \exp[\beta(\mu - \varepsilon_j)^N \tag{11}$$

then

$$P = \frac{NT}{V}\ln(1 + \exp[\beta(\mu - \varepsilon_j) \tag{12}$$

and to calculate the force we can use the fact that:

$$F = AP = L^2\frac{NT}{V}\ln(1 + \exp[\beta(\mu - \varepsilon_j) = \frac{NT}{L}\ln(1 + \exp[\beta(\mu - \varepsilon_j) \tag{13}$$

in which A is the surface of the box side.

(d) The incentive for the particles to still in the mezoscopic box is, the maximum energy should to be small or equal to the minimum energy of the 3D box $\varepsilon_{mezoscopic} \leq \varepsilon_{3D}$,where,$\varepsilon_{mezoscopic} = \varepsilon_{gap} = \frac{3}{2m}(\frac{2\pi}{l})^2$ and $\varepsilon_{3D} = \varepsilon_0 + \varepsilon_P = \frac{\hbar^2k^2}{2m} + mgD$. The D_{min} it's when $\varepsilon_{mezoscopic} = \varepsilon_{3D}$ so we got

$$\frac{3}{2m}(\frac{2\pi}{l})^2 = \frac{\hbar^2k^2}{2m} + mgD_{min} \tag{14}$$

and in the end

$$D_{min} = \frac{1}{2m^2g}[3(\frac{2\pi}{l})^2 - \hbar^2k^2] \tag{15}$$

(e) The incentive for the particles to leave the mezoscopic box is that the minimum energy of 3D box should to be smaller than the maximum energy of the mezoscopic box. Where the temperature was raised so the maximum energy of the mezoscopic box is $E_{max,Mez} = \varepsilon_F + K_BT$ and the minimum energy of the 3D box is $E_{min,3D} = \varepsilon_0 + mgD$.

the condition is :

$$\varepsilon_0 + mgD \leq \varepsilon_F + K_BT \tag{16}$$

To leave the mezoscopic box the particles should have energy bigger than or equal to $E_{max,Mez} = \varepsilon_F + K_BT$. The number of the particles is

$$N' = \int_{E_{max,Mez}}^{\infty} dED(E)f_{FD}(E) \tag{17}$$

where $D(E) = \frac{mL^2l(2m)^{\frac{3}{2}}}{2\pi^2\hbar^3}\sqrt{E - E_{max,Mez}}$

$$f_{FD} = \frac{1}{\frac{1}{z}\exp[\beta E] + 1} \text{ where } z \text{ is the Fugacity } z = \exp[\beta\mu] \tag{18}$$

we defin that $\varepsilon = (E - E_{max,Mez})\beta$.

then:

$$N' = \frac{L^2l(2m)^{\frac{3}{2}}}{\pi^2\hbar^3}\int_0^{\infty} d\varepsilon\frac{\varepsilon^{\frac{1}{2}}}{\frac{1}{z}\exp[\beta\varepsilon] + 1} \tag{22}$$

after subsititution $x = \varepsilon\beta$ and $d\varepsilon = K_BTdx$

$$N' = \frac{L^2l(2mK_BT)^{\frac{3}{2}}}{\pi^2\hbar^3}\int_0^{\infty} dx\frac{x^{\frac{1}{2}}}{\frac{1}{z}\exp[x] + 1} \tag{20}$$

$$N' = \frac{L^2l(2mK_BT)^{\frac{3}{2}}}{\pi^2\hbar^3}0.678 \tag{21}$$