Ex3550: Fermions in Magnetic Field

Submitted by: Alon Ishakbayev

The Problem:

N electrons with mass m and spin $\frac{1}{2}$ are placed in a box at zero temperature $T = 0$. A magnetic field B is applied, such that the interaction is $-\gamma B \sigma_z$ where γ is the gyromagnetic ratio. Consider the following cases:

(a) One dimensional box with length L

(b) Two dimensional box with area A

(c) Three dimensional box with volume V

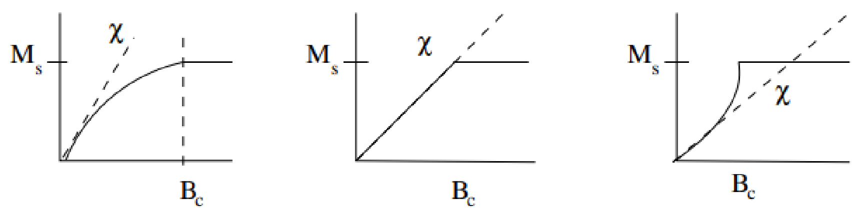
Answer the following questions:

Express your results using γ, m, N, L, A, V.

(1) What is the single particle density of states. Distinguish between a spin up and spin down particles.

(2) Which is the graph that describes the magnetization $M(B)$ of each case (a),(b),(c).

Complete the missing details: what are M_s, B_c ,χ.



The Solution:

(1) By definition the density of states is given by:

$$g(\epsilon) = \frac{\partial N(\epsilon)}{\partial \epsilon} \tag{1}$$

In order to find $g(\epsilon)$, we will find $N(\epsilon)$ the number of states up to energy ϵ

The one particle hamiltonian is given by:

$$H_1 = \frac{\boldsymbol{p}^2}{2m} - \gamma B \sigma_z \tag{2}$$

Therefore the dispersion relation for spin up $\uparrow$ and down $\downarrow$ correspondingly :

$$\epsilon = \frac{\boldsymbol{p}^2}{2m} \mp \gamma B \tag{3}$$

Assuming periodic boundary conditions, momentum states: $\boldsymbol{p} = \frac{\boldsymbol{2\pi}}{\boldsymbol{L}}\boldsymbol{n}$, $(n = 0, \pm1, \pm2...)$

Therefore:

$$|n_{\pm}(\epsilon)| = \frac{L}{2\pi}\sqrt{2m(\epsilon \pm \gamma B)} \tag{4}$$

The total number of states $N(\epsilon)$ for both spins $\uparrow$ and $\downarrow$ correspondingly:

$$N_{\pm}(\epsilon) = \begin{cases} 2n_{\pm}(\epsilon), & 1D \\ \pi n_{\pm}^2(\epsilon), & 2D \\ \frac{4\pi}{3}n_{\pm}^3(\epsilon), & 3D \end{cases} = \begin{cases} \frac{L}{\pi}[2m(\epsilon \pm \gamma B)]^{\frac{1}{2}}, & 1D \\ \frac{A}{2\pi}m(\epsilon \pm \gamma B), & 2D \\ \frac{V}{6\pi^2}[2m(\epsilon \pm \gamma B)]^{\frac{3}{2}}, & 3D \end{cases} \tag{5}$$

Finally, the DOS $g_{\pm}(\epsilon)$:

$$g_{\pm}(\epsilon) = \frac{\partial N(\epsilon)}{\partial \epsilon} = \begin{cases} \frac{L}{2\pi}(2m)^{\frac{1}{2}}(\epsilon \pm \gamma B)^{-\frac{1}{2}}, & 1D \\ \frac{mA}{2\pi}, & 2D \\ \frac{V}{4\pi^2}(2m)^{\frac{3}{2}}(\epsilon \pm \gamma B)^{\frac{1}{2}}, & 3D \end{cases} \tag{6}$$

(2) The magnetization $M(B)$ is given by:

$$M(B) = < -\frac{\partial H}{\partial B} > \tag{7}$$

Total number of electrons at $T = 0$:

$$N = N_{\uparrow} + N_{\downarrow} = \begin{cases} \frac{L}{\pi}\sqrt{2m}[(\epsilon_f + \gamma B)^{\frac{1}{2}} + (\epsilon_f - \gamma B)^{\frac{1}{2}}], & 1D \\ \frac{Am}{\pi}\epsilon_f, & 2D \\ \frac{V}{6\pi^2}(2m)^{\frac{3}{2}}[(\epsilon_f + \gamma B)^{\frac{3}{2}} + (\epsilon_f - \gamma B)^{\frac{3}{2}}], & 3D \end{cases} \tag{8}$$

The hamiltonian of N spins:

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m} + \gamma B(N_{\uparrow} - N_{\downarrow}) \tag{9}$$

The total magnetization $M(B)$:

$$M(B) = \gamma(N_{\uparrow} - N_{\downarrow}) = \begin{cases} \frac{\gamma L}{\pi}\sqrt{2m}[(\epsilon_f + \gamma B)^{\frac{1}{2}} - (\epsilon_f - \gamma B)^{\frac{1}{2}}], & 1D \\ \frac{\gamma^2 Am}{\pi}B, & 2D \\ \frac{\gamma V}{6\pi^2}(2m)^{\frac{3}{2}}[(\epsilon_f + \gamma B)^{\frac{3}{2}} - (\epsilon_f - \gamma B)^{\frac{3}{2}}], & 3D \end{cases} \tag{10}$$

Finding the missing details: M_s, B_c ,χ

From the graphs: above the value B_c all the spins are $\uparrow$, therefore the magnetization M_s for all the three cases is just the number of electrons times the gyromagnetic ratio:

$$M_s = \gamma N \tag{11}$$

To find the critical magnetic field B_c one needs to solve equation (10).
using the fact that $N_\downarrow = 0$, means that the fermi energy is: $\epsilon_f = \gamma B_c$.
Combining equations $(10 + 11)$:

$$M_s = \gamma N = \begin{cases} \frac{\gamma L}{\pi}\sqrt{2m} \cdot (2\gamma B_c)^{\frac{1}{2}}, & 1D \\ \frac{\gamma^2 Am}{\pi} \cdot B_c, & 2D \\ \frac{\gamma V}{6\pi^2}(2m)^{\frac{3}{2}} \cdot (2\gamma B_c)^{\frac{3}{2}}, & 3D \end{cases} \tag{12}$$

Solving equation (12) for B_c:

$$B_c = \begin{cases} \frac{\pi^2 N^2}{4m\gamma L^2}, & 1D \\ \frac{\pi N}{mA\gamma}, & 2D \\ \frac{1}{4m\gamma}(\frac{6\pi^2 N}{V})^{\frac{2}{3}}, & 3D \end{cases} \tag{13}$$

To find χ we will find the slope of $M(B)$ under the condition $B_c \gg B \simeq 0$
The magnetization can be approximated using: $(1+x)^{\frac{1}{2}} \approx 1 + \frac{1}{2}x$, $(1+x)^{\frac{3}{2}} \approx 1 + \frac{3}{2}x$
From (10):

$$M(B) = \begin{cases} \frac{\gamma L}{\pi}\sqrt{2m\epsilon_f}[(1+\frac{\gamma B}{\epsilon_f})^{\frac{1}{2}} - (1-\frac{\gamma B}{\epsilon_f})^{\frac{1}{2}}] \\ \frac{\gamma^2 Am}{\pi} B \\ \frac{\gamma V}{6\pi^2}(2m\epsilon_f)^{\frac{3}{2}}[(1+\frac{\gamma B}{\epsilon_f})^{\frac{3}{2}} - (1-\frac{\gamma B}{\epsilon_f})^{\frac{3}{2}}] \end{cases} \approx \begin{cases} \frac{\gamma L}{\pi}\sqrt{2m}\frac{\gamma}{\sqrt{\epsilon_f}} \cdot B \\ \frac{\gamma^2 Am}{\pi} \cdot B \\ \frac{\gamma^2 V}{2\pi^2}(2m)^{\frac{3}{2}}\sqrt{\epsilon_f} \cdot B \end{cases} \tag{14}$$

Finding ϵ_f around $B \simeq 0$ using equation (8) of the total number of electrons N:

$$N = \begin{cases} \frac{L}{\pi}\sqrt{2m}[(\epsilon_f + \gamma B)^{\frac{1}{2}} + (\epsilon_f - \gamma B)^{\frac{1}{2}}] \\ \frac{Am}{\pi}\epsilon_f \\ \frac{V}{6\pi^2}(2m)^{\frac{3}{2}}[(\epsilon_f + \gamma B)^{\frac{3}{2}} + (\epsilon_f - \gamma B)^{\frac{3}{2}}] \end{cases} \approx \begin{cases} \frac{L}{\pi}\sqrt{2m}2 \cdot \epsilon_f^{\frac{1}{2}} \\ \frac{Am}{\pi} \cdot \epsilon_f \\ \frac{V}{3\pi^2}(2m)^{\frac{3}{2}} \cdot \epsilon_f^{\frac{3}{2}} \end{cases} \tag{15}$$

Therefore:

$$\epsilon_f = \begin{cases} \frac{\pi^2 N^2}{8mL^2} & 1D \\ \frac{\pi N}{Am} & 2D \\ \frac{1}{2m}(\frac{3\pi^2 N}{V})^{\frac{2}{3}} & 3D \end{cases} \tag{16}$$

Substituting (16) into the magnetization from equation (14):

$$M(B) = \begin{cases} \frac{4\gamma^2 L^2 m}{\pi^2 N} \cdot B \\ \frac{M_s}{B_c} \cdot B \\ \frac{3}{2^{\frac{4}{3}}}\frac{M_s}{B_c} \cdot B \end{cases} = \begin{cases} \frac{M_s}{B_c} \cdot B \\ \frac{M_s}{B_c} \cdot B \\ 1.2\frac{M_s}{B_c} \cdot B \end{cases} \tag{17}$$

χ is the linear slope of $M(B)$:

$$\chi = \begin{cases} \frac{M_s}{B_c} & 1D \\ \frac{M_s}{B_c} & 2D \\ 1.2\frac{M_s}{B_c} & 3D \end{cases} \tag{18}$$

To fit each graph to its corresponding case we will use equations $(10)+(8)$ to find $B(M)$ (removing dependence of ϵ_f):

$$\text{B=} \begin{cases} \frac{B_c}{4}[(1+\frac{M}{M_S})^2 - (1-\frac{M}{M_S}^2)] & 1D \\ \frac{B_c}{2}[(1+\frac{M}{M_S}) - (1-\frac{M}{M_S})] & 2D \\ \frac{B_c}{2^{2/3}}[(1+\frac{M}{M_S})^{2/3} - (1-\frac{M}{M_S}^{2/3})] & 3D \end{cases} = \begin{cases} \frac{B_c}{M_S}M, & 1D \\ \frac{B_c}{M_S}M, & 2D \\ \frac{B_c}{2^{2/3}}[(1+\frac{M}{M_S})^{2/3} - (1-\frac{M}{M_S}^{2/3})], & 3D \end{cases} \tag{19}$$

From equation(19) we can see that the linear middle graph represents $1D$ and $2D$ case whereas the left graph represents the $3D$ case.

Results Summary:

Case	Density of States	M_s	B_c	χ	Graph
1D box	$\frac{L}{2\pi}(2m)^{\frac{1}{2}}(\epsilon \pm \gamma B)^{-\frac{1}{2}}$	γN	$\frac{\pi^2 N^2}{4m\gamma L^2}$	$\frac{M_s}{B_c}$	Middle
2D box	$\frac{mA}{2\pi}$	γN	$\frac{\pi N}{mA\gamma}$	$\frac{M_s}{B_c}$	Middle
3D box	$\frac{V}{4\pi^2}(2m)^{\frac{3}{2}}(\epsilon \pm \gamma B)^{\frac{1}{2}}$	γN	$\frac{1}{4m\gamma}(\frac{6\pi^2 N}{V})^{\frac{2}{3}}$	$1.2\frac{M_s}{B_c}$	Left