Ex3550: Fermions in magnetic field - Pauli

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The problem:

N electrons with mass m and spin $\frac{1}{2}$ are placed in a box at zero temperature. A magnetic field B is applied, such that the interaction is $-\gamma B\sigma_z$ where γ is the gyromagnetic ratio. Consider the following cases:

(a) one-dimensional box with length L.

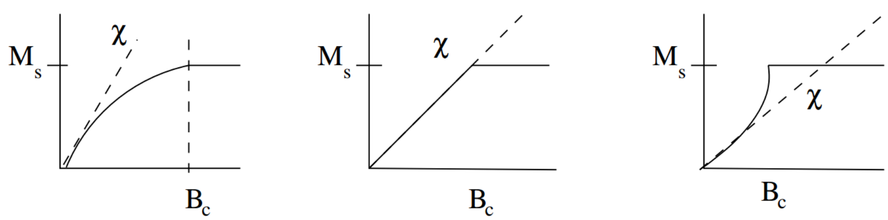
(b) two-dimensional box with area A.

(c) three dimensional box with volume V.

Answer the following questions. Express your results using γ, m, N, L, A, V.

(1) What is the single particle density of states. Distinguish between a spin up and spin down particles.

(2) Which is the graph that describes the magnetization $M(B)$ of each case (a),(b),(c). Complete the missing details: what are M_s, B_c ,χ.



The solution:

(1) The single particle density of states:
In order to get the density of states $g(\epsilon) = \frac{\partial}{\partial\epsilon}\mathcal{N}(\epsilon)$, we need to find $\mathcal{N}(\epsilon)$, the number of states up to energy ϵ. $\mathcal{N}(\epsilon)$ is given by the volume of a d-dimensional ball with radius $\mathbf{n}(\epsilon)$:

$$\mathcal{N}(\epsilon) = \frac{\Omega_d}{d}|\mathbf{n}(\epsilon)|^d,\ \Omega_d = \begin{cases} 2, & d=1 \\ 2\pi, & d=2 \\ 4\pi, & d=3 \end{cases}$$

where $\mathbf{n}(\epsilon)$ is defined from the realtion $\mathbf{p} = \frac{2\pi}{L}\mathbf{n}(\epsilon)$.
For a free particle the dispersion relation is $\epsilon = \frac{p^2}{2m}$, we have $|\mathbf{n}(\epsilon)| = L\frac{\sqrt{2m}}{2\pi}\epsilon^{\frac{1}{2}}$ so that:

$$\mathcal{N}(\epsilon) = \frac{\Omega_d}{d}(L\frac{\sqrt{2m}}{2\pi})^d\epsilon^{\frac{d}{2}} = \frac{1}{d}\mathit{\Lambda}_d\epsilon^{\frac{d}{2}}$$

resulting in:

$$g\left(\epsilon\right)=\frac{1}{2}\Lambda_{d}\epsilon^{\frac{d}{2}-1}$$

In our case, the one-particle Hamiltonian is $H_1 = \frac{p^2}{2m} - \gamma B\sigma_z$, with the dispersion relation $\epsilon = \frac{p^2}{2m} \mp \gamma B$. Using $|\mathbf{n}(\epsilon)| = \frac{L}{2\pi}p = L\frac{\sqrt{2m}}{2\pi}\left(\epsilon \pm \gamma B\right)^{\frac{1}{2}}$, we get the number of states up to energy ϵ for the spin up(+) and down(-) states: $\mathcal{N}_{\pm}\left(\epsilon\right) = \frac{1}{d}\Lambda_d\left(\epsilon \pm \gamma B\right)^{\frac{d}{2}}$.

Finally, we get for the different dimensions:

(a) One-dimensional box with length L: $g_{\pm}(\epsilon) = L\frac{\sqrt{2m}}{2\pi}(\epsilon \pm \gamma B)^{-\frac{1}{2}}$

(b) Two-dimensional box with area A: $g_{\pm}(\epsilon) = A\frac{m}{2\pi}$

(c) Three dimensional box with volume V: $g_{\pm}(\epsilon) = V\frac{(2m)^{\frac{3}{2}}}{4\pi^2}(\epsilon \pm \gamma B)^{\frac{1}{2}}$

(2) We need to use two equations in order to solve this section, the first one relates the total particle and the number of energy states:
At T=0 the particles fills the energy levels up to the fermi energy ϵ_f, so that:

$$N=\mathcal{N}_{+}+\mathcal{N}_{-}=\frac{1}{d}\Lambda_{d}[(\epsilon_{f}+\gamma B)^{\frac{d}{2}}+(\epsilon_{f}-\gamma B)^{\frac{d}{2}}] \tag{1}$$

And the second equation is the total magnetization:
by definition, the one particle magnetization is $M_1 = \langle -\frac{\partial H_1}{\partial B}\rangle = \gamma\langle\sigma_z\rangle$. So that, the total magnetizations is:

$$M=\sum_{i}M_{i}=\gamma(\mathcal{N}_{+}-\mathcal{N}_{-})=\gamma\frac{1}{d}\Lambda_{d}\epsilon_{f}^{\frac{d}{2}}[(1+\frac{\gamma B}{\epsilon_{f}})^{\frac{d}{2}}-(1-\frac{\gamma B}{\epsilon_{f}})^{\frac{d}{2}}] \tag{2}$$

Finding of B_c and M_s:
For $B \geq B_c$ all the particles have spin up with maximal magnetization $M = M_s$. When $B = B_c$ the particles fills the energy levels up to energy $\epsilon_f = \gamma B_c$ (we mention that because N is given ϵ_f isn't a constant as function of B, from eq.(1)). From eq.(1) we get $N = \frac{1}{d}\Lambda_d[(2\gamma B_c)^{\frac{d}{2}} + 0]$, so that:

$$\gamma B_{c}=\frac{1}{2}(\frac{dN}{\Lambda_{d}})^{\frac{2}{d}},\ M_{s}=\gamma N$$

Finding of χ:
For $B \ll B_c$ we can approximate:

$$M\approx\gamma\frac{1}{d}\Lambda_{d}\epsilon_{f}^{\frac{d}{2}}[(1+\frac{d}{2}\frac{\gamma B}{\epsilon_{f}})-(1-\frac{d}{2}\frac{\gamma B}{\epsilon_{f}})]=\gamma\Lambda_{d}\epsilon_{f}^{\frac{d}{2}-1}\gamma B$$

furthermore, we can postulate that $\epsilon_f \approx \epsilon_{f0} + O(B)$, "$\epsilon_{f0}$" is ϵ_f for $B = 0$. for B=0 we will find ϵ_{f0} from eq.(1): $N = \frac{1}{d}\Lambda_d[(\epsilon_{f0})^{\frac{d}{2}} + (\epsilon_{f0})^{\frac{d}{2}}] \Rightarrow\ \epsilon_{f0} = (\frac{dN}{2\Lambda_d})^{\frac{2}{d}} = 2^{1-\frac{2}{d}}\gamma B_c$, for this postulation we get:

$$M\approx\frac{d}{2^{2-\frac{2}{d}}}\frac{\gamma N}{B_{c}}B\equiv\chi B$$

In order to relate the graphs to the different systems we need to compare the slope of $M = \chi B$ with the slope of the straight line $M = \frac{M_s}{B_c}B$. We can see that in the left graph $\chi > \frac{M_s}{B_c}$, in the right one

$\chi < \frac{M_s}{B_c}$, and in the middle one $\chi = \frac{M_s}{B_c}$. So that, the equation $\chi = \frac{d}{2^{2-\frac{2}{d}}}\frac{M_s}{B_c}$ relates the graphs and the dimensions.

Finnaly we have:
for all the three cases $M_s = \gamma N$.

(a) One-dimensional box with length L: $\gamma B_c = \frac{\pi^2 N^2}{4mL^2}$, $\chi = \frac{M_s}{B_c}$ - this fits the middle graph.

(b) Two-dimensional box with area A: $\gamma B_c = \frac{\pi N}{Am}$, $\chi = \frac{M_s}{B_c}$ - this fits the middle graph too.

(c) Three dimensional box with volume V: $\gamma B_c = \frac{1}{4m}(\frac{6\pi^2 N}{V})^{\frac{2}{3}}$, $\chi = \frac{3}{2^{2-\frac{2}{3}}}\frac{M_s}{B_c}$ - this fits the left graph.

Appendix: Drawing the graphs.

In order to draw the graphs we need to find $M(B)$, but actually it is easier to find $B(M)$. We want to discard ϵ_f. From eq.(1-2) we can get:

$$\epsilon_f + \gamma B = [\frac{dN}{2\Lambda_d}(1 + \frac{M}{\gamma N})]^{\frac{2}{d}}$$

$$\epsilon_f - \gamma B = [\frac{dN}{2\Lambda_d}(1 - \frac{M}{\gamma N})]^{\frac{2}{d}}$$

so that the equation that relates M and B is:

$$B = 2^{-\frac{2}{d}} B_c[(1 + \frac{M}{M_s})^{\frac{2}{d}} - (1 - \frac{M}{M_s})^{\frac{2}{d}}]$$

for the approximation of $M \ll M_s$ we can get χ again:

$$B = 2^{-\frac{2}{d}} B_c \frac{4}{d}\frac{M}{M_s} \;\Rightarrow\; M = \frac{d}{2^{2-\frac{2}{d}}}\frac{M_s}{B_c}B = \chi B$$