E3520: Ideal Fermi gas in 2D space

Submitted by: Amit Omer and Naor Yossi

The problem:

N fermions with mass m and spin $\frac{1}{2}$ are in a two dimensional box with area A.

1) Show that: $N(\beta,\mu) = A\frac{\mathsf{m}}{\pi}T\ln\left(1+e^{\frac{\mu}{T}}\right)$

Guideline: Use variables exchange $X = e^{\beta(E-\mu)}$ and use the integral $\int_1^\infty \frac{dx}{x(x+1)} = \ln\left(\frac{1+X_1}{X_1}\right)$.

2) Take $T = 0$ and find $N = A\frac{m}{\pi}\mu$. Explain that result intuitively by looking at $g(E)$.

3) Find the chemical potential $\mu\left(T, \frac{N}{A}\right)$ and the fermi energy $E_F \equiv \mu(T \to 0)$.

4) Show that in low temperatures

$$\mu(T) \approx E_F - Te^{-\frac{E_F}{T}}$$

While in high temperatures we get a result consistant with the general expression you developed in the Boltzmann's proximity frame.

5) Find $E(\beta,\mu)$ and $P(\beta,\mu)$ in temperature zero, and by placing the expression for $\mu\left(T=0, \frac{N}{A}\right)$. Show that the following results

$$E = A\frac{\pi}{m}\frac{1}{2}\left(\frac{N}{A}\right)^2, P = \frac{\pi}{m}\frac{1}{2}\left(\frac{N}{A}\right)^2$$

6) Clarify yourself why in zero temperature $P \propto \frac{1}{A^2}$ while in high temperature $P \propto \frac{1}{A}$.

The solution:

1)

$$Z=\sum_{\{n_i\}}e^{-\beta(E_i-\mu n_i)}$$

$$E_i=\epsilon_i\cdot n_i,\quad n_i=0,1\ -\ fermions,\ K_B=1\ \left(\beta=\frac{1}{T}\right)$$

$$Z=\sum_{\{x_i,p_i,\sigma_i,n_i\}}e^{-\beta\cdot n_i(\epsilon_i-\mu)}=\prod_{\{x_i,p_i,\sigma_i\}}\left(1+e^{-\beta(\epsilon_i-\mu)}\right)$$

$$N=T\frac{\partial\ln Z}{\partial\mu}=\sum_{\{x_i,p_i,\sigma_i\}}\frac{1}{1+e^{\beta(\epsilon_i-\mu)}}$$

$$\epsilon_i=\frac{p_i^2}{2m},\quad spin\frac{1}{2}\ \rightarrow\ degeneracy\ 2\ \ (no\,external\,magnetic\,field)$$

$$=\frac{2}{(2\pi)^2}\cdot\int_A dA\int_0^{2\pi}d\phi\int_0^\infty\frac{pdp}{1+e^{\beta\left(\frac{p^2}{2m}-\mu\right)}}$$

$$x=e^{\beta\left(\frac{p^2}{2m}-\mu\right)}$$

$$=\frac{Am}{\pi\beta}\int_{e^{-\beta\mu}}^\infty\frac{dx}{x\,(1+x)}=\frac{Am}{\pi\beta}\ln\left(\frac{1+e^{-\beta\mu}}{e^{-\beta\mu}}\right)=\frac{AmT}{\pi}\ln\left(1+e^{\frac{\mu}{T}}\right)$$

2)

$$T=0\ \rightarrow\ e^{\frac{\mu}{T}}\gg 1$$

$$N=\frac{AmT}{\pi}\ln\left(1+e^{\frac{\mu}{T}}\right)=\frac{AmT}{\pi}\ln\left(e^{\frac{\mu}{T}}\right)=\frac{Am\mu}{\pi}$$

We can look at it intuitively by saying that at T=0 all the energy levels are populated equaly (2 fermions at each level) until E_F, so that the density number is just constant.

for ideal fermions gas, spin $\frac{1}{2}$ and 2D: $g(E)=\frac{mA}{\pi}$

(according to section 6.2 in the lecture notes)

so $N=g\cdot E_F|_{T\to 0}=\frac{Am\cdot\mu}{\pi}$ (as we got in the direct way).

3)

$$N = \frac{AmT}{\pi}\ln\left(1+e^{\frac{\mu}{T}}\right) \quad \rightarrow \quad \mu(T,\frac{N}{A}) = T\cdot\ln\left(e^{\left(\frac{N}{A}\right)\cdot\frac{\pi}{mT}}-1\right)$$

$$E_F \equiv \mu\left(T\rightarrow 0\right)$$

$$e^{\left(\frac{N\pi}{AmT}\right)}\left[T\rightarrow 0\right] \gg 1$$

$$\mu(T,\frac{N}{A}) = T\cdot\ln\left(e^{\left(\frac{N\pi}{AmT}\right)}\right) = \frac{N\pi}{Am}$$

$$\mu\left(T\rightarrow 0\right) \equiv E_F = \frac{N\pi}{Am}$$

4)

<u>Low Temperatures:</u>

$$\frac{N\pi}{AmT} = \frac{E_F}{T} = \ln\left(1+e^{\frac{\mu}{T}}\right)$$

$$\mu = T\cdot\ln\left(e^{\left(\frac{E_F}{T}\right)}-1\right) = T\cdot\ln\left[e^{\left(\frac{E_F}{T}\right)}\cdot\left(1-e^{\left(\frac{-E_F}{T}\right)}\right)\right]$$

$$= T\left[\frac{E_F}{T}+\ln\left(1-e^{\left(\frac{-E_F}{T}\right)}\right)\right]$$

$$e^{\left(\frac{-E_F}{T}\right)}\left[T\rightarrow 0\right]\rightarrow 0 \quad \text{and as we know} \quad \lim_{x\rightarrow 0}\ln\left(1+x\right)\approx x$$

therefore

$$\mu \approx T\left[\frac{E_F}{T}-e^{\left(\frac{-E_F}{T}\right)}\right] = E_F - T\cdot e^{\left(\frac{-E_F}{T}\right)}$$

High Temperatures:

$$n\lambda_T^2 \ll 1 \quad therefore \quad e^{\frac{\mu}{T}} \ll 1$$

$$N = \frac{AmT}{\pi} \ln\left(1 + e^{\frac{\mu}{T}}\right) \approx \frac{AmT}{\pi} \cdot e^{\frac{\mu}{T}}$$

$$\mu = T \cdot \ln\left(\frac{\pi N}{AmT}\right)$$

In the Boltzmann proximity frame $\mu = T \cdot \ln\left(\frac{N\lambda_T^2}{Ag_0}\right)$

$$g_0 = \sum_{spin} e^{-\beta\epsilon_{bound}} = 2, \quad (no\, external\, magnetic\, field)$$

$$\lambda_T^2 = \left(\frac{2\pi}{mT}\right)$$

$$\mu = T \cdot \ln\left(\frac{2N\pi}{AmT} \cdot \frac{1}{2}\right) = T \cdot \ln\left(\frac{\pi N}{AmT}\right)$$

as we can see the result we derived is consistant with the Boltzmann proximity frame.

5)

according to section 6.2 in the lecture notes:

$$\frac{E}{A} = \frac{m}{\pi} T^2 F_2(z)$$

for $T \to 0$ we get $z \gg 1$ and $F_2 \approx \frac{1}{2}\left[ln(z)\right]^2$

$$\ln(z) = \ln\left(e^{\frac{\mu}{T}}\right) = \frac{\mu(T \to 0)}{T} = \frac{E_F}{T} = \frac{\pi N}{AmT}$$

$$\frac{E}{A} \approx \frac{m}{2\pi} T^2 \left(\frac{\mu}{T}\right)^2 = \frac{m}{2\pi} T^2 \left(\frac{\pi N}{AmT}\right)^2$$

$$E = \frac{\pi A}{2m} \left(\frac{N}{A}\right)^2$$

At $T = 0$ we get $E = F - TS = F$ therefore $P = -\left(\frac{\partial F}{\partial A}\right)_N = -\left(\frac{\partial E}{\partial A}\right)_N$

$$P = \frac{\pi}{2m} \left(\frac{N}{A}\right)^2$$

For high temperatures we have ideal gas in Boltzmann approximation so that simply $P = \frac{NT}{A}$

6)

The persure reflects the Dependence of the energy levels on the linear size of the box.

$P = -\frac{\partial E_n}{\partial A}$ averaged over n.

$E_n \propto \frac{n^2}{L^2} = \frac{n^2}{A}$ hence $P \propto \frac{n^2_{typical}}{A^2}$.

for N fermion at zero temperature the typical value is $n \sim N$, therefore $P \propto \frac{1}{A^2}$.

for high temperature T the typical value is determined from $E_n \sim T$ i.e.

$n^2 \sim TA$ and therefore $P \propto \frac{TA}{A^2} \propto \frac{1}{A}$.