Ex3515: Ideal Fermi gas in 1D space

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The problem:

Consider N electrons that are kept between the plates of a capacitor.

$$V(x,y,z) = \begin{cases} \frac{1}{2}m\omega^2\left(x^2+y^2\right) & 0 \le z \le L \\ \infty & \text{else} \end{cases}$$

The system is in thermal equilibrium at zero temperature. Find the force that the gas exerts of the plates assuming that it can be treated as one-dimensional.
Write the condition on N for having this assumption valid.
Tip: Find first the one particle states, and illustrate them using a schematic drawing. Express your results using N, L, m, ω only.

The solution:

The one-particle energy spectrum is given by

$$\epsilon_{n_x,n_y}(p_z) = \omega(n_x+n_y+1) + \frac{p_z^2}{2m} \qquad (\hbar = 1) \tag{1}$$

Where $p_z = \frac{\pi}{L}n_z$.

It is assumed that in the range where the system can be treated as one-dimensional, the contribution of energy levels $n_x, n_y > 0$ can be ignored, i.e.

$$\epsilon = \omega + \frac{p_z^2}{2m} \tag{2}$$

The one-particle energy range for which this limit is valid, is then anywhere between the ground state, and the 1st excited state on the XY plane

$$\omega < \epsilon < 2\omega \tag{3}$$

The occupation of particles in this energy range occurs over the $\epsilon_{n_z} = \frac{1}{2m}\left(\frac{\pi}{L}n_z\right)^2$ levels, generally ranging from 0 to ω.

At T=0, the Fermi distribution behaves like a step function, with energy states lower than $\epsilon_F = \mu$.
The relation between N and the chemical potential μ is then given by

$$N = \sum_i \langle n_i \rangle = 2\int \frac{dz\, dp_z}{2\pi} = \frac{2L}{\pi}\sqrt{2m}\,(\mu-\omega)^{1/2} \tag{4}$$

Combining (3) & (4), (with $\omega < \epsilon_F < 2\omega$), gives a condition on N for the validity of this limit

$$N < \frac{2L}{\pi}\sqrt{2m\omega} \tag{5}$$

At a general temperature T, the force P exerted on the plates can generally be found by calculating the grand canonical potential Ω, and by using the relation $PL = -\Omega$.
At T=0 it is also possible to find P through the total energy of the system, which is given by

$$E = \sum_i \epsilon_i \langle n_i \rangle = 2\int \frac{dz\, dp_z}{2\pi}\,\epsilon(p_z) = \omega N + \frac{1}{3}\left(\frac{\pi}{2L\sqrt{2m}}\right)^2 N^3 \tag{6}$$

The force exerted on the plates is then

$$P = -\frac{\Omega}{L} = -\frac{(E-\mu N)}{L} = -\left(\frac{\partial E}{\partial L}\right)_N = \frac{\pi^2}{12m}\left(\frac{N}{L}\right)^3 \tag{7}$$