Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

[Exercise 3344]

BEC in harmonic potential

The current experimental realizations of Bose Einstein condensation rely on trapping cold atoms in a potential. Close to its minimum, the potential can be expanded to second order, and has the form

$$U(\vec{r}) = \tfrac{1}{2}m\sum_{\alpha}\omega_\alpha^2 x_\alpha^2$$

where $\alpha = 1, ..., d$, d is the space dimensionality and the trapping potential may have different frequencies ω_α in different directions.

(a) We are interested in the limit of wide traps such that $\hbar\omega_\alpha \ll k_B T$, and the discreteness of the allowed energies can be largely ignored. Show that in this limit, the number of states $N(E)$ with energy less than or equal to E, and the density of states $\rho(E) = dN(E)/dE$ are given by

$$N(E) = \frac{1}{d!}\prod_{\alpha=1}^{d}\left(\frac{E}{\hbar\omega_\alpha}\right) \qquad \Rightarrow \qquad \rho(E) = \frac{1}{(d-1)!}\frac{E^{d-1}}{\prod_{\alpha=1}^{d}\hbar\omega_\alpha}$$

[Hint: The volume of the hyper-pyramid defined by $\sum_{i=1}^{d} x_i \leq R$ and $x_i \geq 0$, in d dimensions is $R^d/d!$.]

(b) Show that in a grand canonical ensemble, the number of particles in the trap is

$$\langle N\rangle = g_d(\zeta)\prod_{\alpha=1}^{d}\left(\frac{k_B T}{\hbar\omega_\alpha}\right)$$

where $g_n(\zeta)$ is the usual Bose function.

(c) Find the chemical potential in the high temperature limit.

(d) Find the temperature T_c for BE condensation (no need to evaluate the g_d integrals). At which dimensions there is no solution with finite T_c?

[Note that the condensate is confined by the trap to a finite size so that the system does not have a proper thermodynamic ($N \to \infty$) limit. Nonetheless, there is a reasonable sharp crossover temperature T_c, at which a macroscopic fraction of particles condenses to the ground state.]