Ex1320: Bose in 2D harmonic trap

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The problem:

Consider a two dimensional bose gas in a harmonic potential with energy eigenvalues $\hbar\omega(n_1+n_2+1)$ where n_1, n_2 are integers. [This is how the recent cold atom experiments realize condensation].

(1) Find the number of particles in excited states $N - N_0$, where N_0 is the the ground state occupation. Assume $k_B T \gg \hbar\omega$ so that summations on n_1, n_2 can be replaced by integrals.
(2) Find the Bose Einstein condensation temperature T_c. [Note that N here is not taken to its thermodynamic limit; the transition is still fairly sharp if $N >> 1$.]
* use the integral - $\int_0^\infty \frac{x}{e^x-1}dx = \frac{\pi^2}{6}$
(3) Show that for $T < T_c$: $N_0 = N[1-(T/T_c)^2]$.

The solution:

(1) The partition function in the Grand- canonical ensemble is:

$$Q = \sum_{n_1,n_2=1}^{\infty}\sum_{N_\epsilon}^{\infty} e^{-\beta N_\epsilon(\epsilon_{n_1,n_2}-\mu)} = \sum_{n_1,n_2=1}^{\infty} \frac{1}{1-e^{-\beta(\epsilon_{n_1,n_2}-\mu)}}$$

The total occupation number is : $<n>= \frac{1}{\beta}\frac{\partial \ln Q}{\partial \mu}$,

$$<n>= \sum_{n_1,n_2=1}^{\infty} \frac{e^{-\beta(\epsilon_{n_1,n_2}-\mu)}}{1-e^{-\beta(\epsilon_{n_1,n_2}-\mu)}} = \sum_{n_1,n_2=1}^{\infty} \frac{1}{e^{+\beta(\epsilon_{n_1,n_2}-\mu)}-1}$$

In order to replace the summation to integral over the energies , we should find first the number of states up to energy E , and the density of states:

The relation between n_1, n_2 and the energy is describe by the equation: $\epsilon = \hbar\omega(n_1+n_2+1)$
Creating a right triangle at the plane of n_1, n_2 with leg length of $\frac{\epsilon}{\hbar\omega}$ so:

$$N(\epsilon) = \frac{\epsilon^2}{2\hbar\omega^2} \text{ , } g(\epsilon) = \frac{\partial N(\epsilon)}{\partial \epsilon} = \frac{\epsilon}{(\hbar\omega)^2}$$

Having the density of states, we can replace the summation to an integral of ϵ, we should notice that by writing it as an integral, we count only the occupation of the excited states.

$$N_{excited} = \int_0^\infty \frac{1}{e^{+\beta(\epsilon-\mu)}-1} g(\epsilon) d\epsilon = \frac{1}{(\hbar\omega)^2}\int_0^\infty \frac{\epsilon}{e^{-\beta\mu}\cdot e^{\beta\epsilon}-1}d\epsilon$$

We shall write it in the manner of polylogarithm functions , which can be defined as : $g_\nu(Z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x}{z^{-1}e^x-1}dx$
Replacing $\beta\epsilon = x$ we get :

$$N - N_0 = N_{excited} = \frac{T^2}{(\hbar\omega)^2}\cdot g_2(e^{\beta\mu})\cdot\Gamma(2) = \frac{T^2}{(\hbar\omega)^2}\cdot g_2(e^{\beta\mu})$$

(2) We are asked to find the critical temperature of the Bose -Einstein condensation , so we just have to ask on which temperature the number of particles occupied on the excited states is equal to

the total number of particles N, $g_2(e^{\beta\mu})$ can be taken to be $g_2(1) = \int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}$ when $\mu << T$ which is something seems to characterize the condensation - lower μ means there is no energy cost in adding another particle to the system

$$N_{excited} = \frac{\pi^2 \cdot T^2}{6 \cdot (\hbar\omega)^2} = N \to T_c = \pi\hbar\omega \cdot \sqrt{\frac{N}{6}}$$

(3) Beneath the critical temperature, the occupation of the ground state is equal to $N - N_{excited}$:

$$N_0 = N - N_{excited} = N - \frac{\pi^2 \cdot T^2}{6 \cdot (\hbar\omega)^2} = N - \frac{\pi^2}{6 \cdot (\hbar\omega)^2}(T/T_c)^2 \cdot T_c^2 = N(1 - (T/T_c)^2)$$