3336: Condesation for general dispersion

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The problem:

An ideal Bose gas whose single particle spectrum is given by $\epsilon = C \cdot |p|^s, s > 0$ is contained in volume V of dimention d. The gas is in uniform temperature T.

(1) Calculate the one-particle density of states.

(2) Find the condition on s and d for the existence of Bose-Einstein condensation. In particular relate to relativistic (s=1) and nonrelativistic (s=2) particles in two dimensions (d=2).

(3) Find the dependence of the number of particles N on the chemical potential μ.

(4) Find the dependence of the total energy E on the chemical potential, and show how the pressure P is obtained from this result.

(5) Find an expression for the heat capacity C_v as a function of N in the limit of infinite temperature.

(6) Repeat item (1) for relativistic gas whose particles have finite mass such that their dispersion relation is $\epsilon = \sqrt{m^2c^4 + c^2p^2}$.

(7) Consider a relativistic gas in $2D$. Find expressions for N and E and P. Should one expect Bose-Einstein condensation?

The solution:

(1)
The particle spectrum is given by $\epsilon = C \cdot |p|^s \Leftrightarrow |p|^2 = (\epsilon/C)^{2/s}$

$$\nu(\epsilon) = \int\int_{H<E} \frac{d^dxd^dp}{(2\pi)^d} = \frac{V}{(2\pi)^d}\int\int_{k_1^2+...+k_d^2<(\epsilon/C)^{2/s}} d^dk = \frac{V}{(2\pi)^d}\cdot\Omega_d\int_0^k k^{d-1}dk = \frac{V}{(2\pi)^d}\frac{\Omega_d}{d}k^d$$

$$k^d = (k^2)^{d/2} = ((\epsilon/C)^{2/s})^{d/2} = (\epsilon/C)^{d/s}$$

and therefore

$$\nu(\epsilon) = \frac{s}{d}\frac{V}{s\cdot(2\pi)^d}\frac{\Omega_d}{C^{d/s}}\epsilon^{d/s}$$

$$g(\epsilon) = \frac{\partial\nu}{\partial\epsilon} = \frac{V}{s\cdot(2\pi)^d}\frac{\Omega_d}{C^{d/s}}\epsilon^{d/s-1}$$

(2)
We define $\alpha = \frac{d}{s}$ and $c = \frac{1}{s\cdot(2\pi)^d}\frac{\Omega_d}{C^{d/s}}$

$$N(\epsilon) = V\cdot c\cdot\int_0^\infty d\epsilon\frac{\epsilon^{\alpha-1}}{e^{\beta(\epsilon-\mu)}-1}$$

When we have BEC we get $\mu \to 0$.
In order to exhibit BEC the integral should converge and so for $\mu \to 0$ the integrant should fulfill a condition that $\alpha > 1 \Leftrightarrow d > s$.
For 2D nonrelativistic particle $d = s = 2$. From above the system will not exhibit BEC.
For 2D relativistic particle (with rest mass $m = 0$) we have $d = 2$ and $s = 1$. From the above the particles can exhibit BEC.

(3)
We define $z = e^{\beta\mu}$

$$N(\epsilon) = V \cdot c \cdot \int_0^\infty d\epsilon \frac{\epsilon^{\alpha-1}}{\frac{1}{z}e^{\beta\epsilon} - 1} \underbrace{=}_{x=\frac{\epsilon}{T}} V \cdot c \cdot T^\alpha \cdot \int_0^\infty dx \frac{x^{\alpha-1}}{\frac{1}{z}e^x - 1} = V \cdot c \cdot T^\alpha \cdot \Gamma(\alpha) \cdot F_\alpha(z)$$

where

$$F_\alpha(z) = \sum_{l=1}^\infty \frac{\pm(\pm 1)^l}{l^\alpha} z^l = \sum_{l=1}^\infty \frac{\pm(\pm 1)^l}{l^\alpha} e^{l\beta\mu}$$

$$\Rightarrow N(\mu) = V \cdot c \cdot T^\alpha \cdot \Gamma(\alpha) \cdot \sum_{l=1}^\infty \frac{\pm(\pm 1)^l}{l^\alpha} e^{l\beta\mu}$$

(4)

$$E(\epsilon) = V \cdot c \cdot \int_0^\infty d\epsilon \frac{\epsilon^{\alpha-1}\epsilon}{\frac{1}{z}e^{\beta\epsilon} - 1} \underbrace{=}_{x=\frac{\epsilon}{T}} V \cdot c \cdot T^{\alpha+1} \cdot \int_0^\infty dx \frac{x^\alpha}{\frac{1}{z}e^x - 1} \underbrace{=}_{\nu=\alpha+1} V \cdot c \cdot T^{\alpha+1} \cdot \int_0^\infty dx \frac{x^{\nu-1}}{\frac{1}{z}e^x - 1}$$

And we get

$$E(\epsilon) = V \cdot c \cdot T^{\alpha+1} \cdot \Gamma(\nu) \cdot F_\nu(z) = V \cdot c \cdot T^{\alpha+1} \cdot \Gamma(\alpha+1) \cdot F_{\alpha+1}(z)$$

we should also notice that

$$E(\epsilon) = \frac{\alpha}{\beta} lnZ$$

We know that

$$P = \frac{1}{\beta}\frac{lnZ}{V} = \frac{1}{\alpha}\frac{E}{V} = \frac{c}{\alpha} \cdot T^{\alpha+1} \cdot \Gamma(\alpha+1) \cdot F_{\alpha+1}(z) \underbrace{=}_{\alpha=\frac{d}{s}} \frac{s}{d}\frac{E}{V}$$

(5)
The limit of infinite temperature is Boltzman approximation.
For Boltzman approximation we get:

$$E(\epsilon) = V \cdot c \cdot \int_0^\infty d\epsilon \frac{\epsilon^{\alpha-1}\epsilon}{\frac{1}{z}e^{\beta\epsilon}} = V \cdot c \cdot T^{\alpha+1} \cdot \Gamma(\alpha+1) \cdot z = V \cdot c \cdot T^{\alpha+1} \cdot \alpha \cdot \Gamma(\alpha) \cdot z$$

and

$$N(\epsilon) = V \cdot c \cdot \int_0^\infty d\epsilon \frac{\epsilon^{\alpha-1}}{\frac{1}{z}e^{\beta\epsilon}} = V \cdot c \cdot T^\alpha \cdot \Gamma(\alpha) \cdot z$$

$$\Rightarrow E = \alpha \cdot N \cdot T$$

$$C_v = \frac{\partial E}{\partial T}|_{V,N} = \alpha \frac{\partial (NT)}{\partial T}|_{V,N} = \alpha N = \frac{d}{s} N$$

(6)

$$\epsilon = \sqrt{m^2c^4 + c^2|p|^2} \Rightarrow |p|^2 = (\frac{\epsilon}{c})^2 - m^2c^2$$

as we did in (1)

$$\nu(\epsilon) = \int\int_{H<E} \frac{d^dxd^dp}{(2\pi)^d} = \frac{V}{(2\pi)^d} \int\int_{k_1^2+...+k_d^2<(\frac{\epsilon}{c})^2-m^2c^2} d^dk = \frac{V}{(2\pi)^d} \cdot \Omega_d \int_0^k k^{d-1}dk = \frac{V}{(2\pi)^d} \frac{\Omega_d}{d} k^d$$

$$k^d = (k^2)^{d/2} = ((\epsilon/c)^2 - m^2c^2)^{d/2}$$

and therefore

$$\nu(\epsilon) = \frac{1}{d} \frac{V}{(2\pi)^d} \Omega_d((\frac{\epsilon}{c})^2 - m^2c^2)^{d/2}$$

$$g(\epsilon) = \frac{\partial \nu}{\partial \epsilon} = \frac{V}{(2\pi)^d} \Omega_d((\frac{\epsilon}{c})^2 - m^2c^2)^{d/2-1} \cdot \frac{\epsilon}{c^2}$$

(7)
We consider $2D$ relativistic gas therefore

$$N(\epsilon) = \Sigma_k f(\epsilon_k - \mu) = \frac{V}{(2\pi)^2} \cdot \int_0^\infty 2\pi k dk \frac{1}{e^{\beta(\sqrt{k^2c^2+m^2c^4}-\mu)} - 1}$$

we take $\epsilon = \sqrt{k^2c^2 + m^2c^4} \Rightarrow \frac{d\epsilon}{dk} = \frac{kc^2}{\sqrt{k^2c^2+m^2c^4}} \Rightarrow \epsilon d\epsilon / c^2 = kdk$ and get

$$N(\mu) = \frac{V}{2\pi c^2} \cdot \int_{mc^2}^\infty \epsilon d\epsilon \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \underbrace{=}_{\epsilon^{`}=\epsilon-mc^2} \frac{V}{2\pi c^2} \cdot \int_0^\infty d\epsilon^{`} \frac{\epsilon^{`} + mc^2}{e^{\beta(\epsilon^{`}+mc^2-\mu)} - 1}$$

$$= \frac{V}{2\pi c^2} \cdot (\int_0^\infty d\epsilon^{`} \frac{\epsilon^{`}}{e^{\beta(\epsilon^{`}+mc^2-\mu)} - 1} + \int_0^\infty d\epsilon^{`} \frac{mc^2}{e^{\beta(\epsilon^{`}+mc^2-\mu)} - 1}) = \frac{V}{2\pi c^2} \cdot (k_b^2T^2\Gamma(2)F_2(z^{`}) + mc^2k_bT\Gamma(1)F_1(z^{`}))$$

where $z^{`} = e^{\beta(\mu-mc^2)}$ and $F_1(z^{`}) = -ln(1-z^{`})$.
The same way

$$E(\epsilon) = \Sigma_k \epsilon f(\epsilon_k - \mu) = \frac{V}{(2\pi)^2} \cdot \int_0^\infty 2\pi k dk \frac{\epsilon}{e^{\beta(\sqrt{k^2c^2+m^2c^4}-\mu)} - 1} = \frac{V}{2\pi c^2} \cdot \int_0^\infty d\epsilon^{`} \frac{(\epsilon^{`} + mc^2)^2}{e^{\beta(\epsilon^{`}+mc^2-\mu)} - 1}$$

$$\Rightarrow E(\mu) = \frac{V}{2\pi c^2} \cdot (k_b^3T^3\Gamma(3)F_3(z^{`}) + 2mc^2k_b^2T^2\Gamma(2)F_2(z^{`}) + (mc^2)^2k_bT\Gamma(1)F_1(z^{`}))$$

We want to calculate $P = \frac{1}{\beta} \frac{lnZ}{V}$

$$lnZ = -\Sigma_k(ln(1 - e^{-\beta(\epsilon_k-\mu)})) = -\frac{V}{(2\pi)^2} \cdot \int_0^\infty 2\pi k dk ln(1 - e^{-\beta(\sqrt{k^2c^2+m^2c^4}-\mu)})$$

doing integration by parts and taking $\epsilon = \sqrt{k^2c^2 + m^2c^4}$ as before we get

$$lnZ = \frac{V\beta}{2\pi} \cdot \int_{mc^2}^\infty d\epsilon((\frac{\epsilon}{c})^2 - m^2c^2) \frac{1}{e^{\beta(\epsilon-\mu)} - 1} = \frac{V\beta}{2\pi c^2} \cdot \int_0^\infty d\epsilon^{`} \frac{(\epsilon^{`} + mc^2)^2 - m^2c^4}{e^{\beta(\epsilon^{`}+mc^2-\mu)} - 1}$$

therefore

$$P(\mu) = \frac{1}{\beta}\frac{lnZ}{V} = \frac{1}{2\pi c^2}\cdot\int_0^\infty d\epsilon^{`}\frac{(\epsilon^{`}+mc^2)^2-m^2c^4}{e^{\beta(\epsilon^{`}+mc^2-\mu)}-1} = \frac{1}{2\pi c^2}\cdot(k_b^3T^3\Gamma(3)F_3(z^{`})+2mc^2k_b^2T^2\Gamma(2)F_2(z^{`}))$$

Now we want to consider weather the system can exhibit BEC.

$$N(\mu) = \frac{V}{2\pi c^2}\cdot\int_0^\infty d\epsilon^{`}\frac{\epsilon^{`}+mc^2}{e^{\beta(\epsilon^{`}+mc^2-\mu)}-1}\underbrace{=}_{\mu^{`}=\mu-mc^2}\frac{V}{2\pi c^2}\cdot\int_0^\infty d\epsilon^{`}\frac{\epsilon^{`}+mc^2}{e^{\beta(\epsilon^{`}-\mu^{`})}-1}$$

For $\mu^{`}=0 \Leftrightarrow \mu = mc^2$ the integral does not converge and so the system can‘t get BEC.