E3230: Heat Capacity of He4 system, energy gap

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The problem:

The specific heat of He^4 at low temperatures has the form

$$C_v = A(T) + B(T)e^{-\Delta/T}$$

This is explained by having a dispersion relation that give rise to long wavelength phonons, $\omega = c|\boldsymbol{k}|$ and short wavelength rotons $\omega(k) = \Delta + b(|\boldsymbol{k}| - k_0)^2$, where $k_0 = 1/a$ is comparable to the mean interparticle separation.

(a) Find explcit expressions for the coefficients $A(T)$ and $B(T)$

(b) What would be the power in the T dependence of the coefficients if the system was two dimensional?

The solution:

(a) For phonons: $\varepsilon = \omega$

$$k = \frac{\omega}{c}$$

$$f(\omega) = \frac{1}{e^{\frac{\omega}{T}} - 1}$$

$$E(x,k) = \int \frac{d^3x d^3k}{(2\pi)^3}\varepsilon(k)f(k)$$

$$E(\omega) = \frac{4\pi V}{(2\pi)^3}\int \omega f(\omega)k^2(\omega)dk(\omega) = \frac{V}{2\pi^2}\int_0^\infty \frac{\omega^2}{c^2}\frac{1}{e^{\frac{\omega}{T}} - 1}\omega d\left(\frac{\omega}{c}\right)$$

$$E(\omega)_{phonons} = \frac{VT^4}{2\pi^2 c^3}\int_0^\infty \left(\frac{\omega}{T}\right)^3 \frac{1}{e^{\frac{\omega}{T}} - 1} d\left(\frac{\omega}{T}\right) = \frac{VT^4}{2\pi^2 c^3}\left(\frac{\pi^4}{15}\right)$$

The dispersion relation for rotons: $\varepsilon = \omega(k) = \Delta + b(|\boldsymbol{k}| - k_0)^2$

$$E(k) = \frac{4\pi V}{(2\pi)^3}\int \omega(k)f(k)k^2 dk$$

For short wavelength rotons at low temperatures, we take $\frac{\omega}{T} >> 1$ and so $\frac{1}{e^{\frac{\omega}{T}}-1} \approx e^{-\frac{\omega}{T}}$

$$E(k) = \frac{V}{2\pi^2}\int_0^\infty (\Delta + b(|\boldsymbol{k}| - k_0)^2)e^{-\frac{\Delta + b(|\boldsymbol{k}| - k_0)^2}{T}}k^2 dk$$

Denote $x = \sqrt{\frac{b}{T}}(|\boldsymbol{k}| - k_0)$

$$E(T) = \frac{V}{2\pi^2}\int_0^\infty \left(\Delta + Tx^2\right)e^{-(\frac{\Delta}{T} + x^2)}\left(\sqrt{\frac{T}{b}}x + k_0\right)^2\sqrt{\frac{T}{b}}dx$$

$$E(T) = \frac{Vk_0^2}{2\pi^2}\sqrt{\frac{T}{b}}e^{-\frac{\Delta}{T}}\int_0^\infty \left(\frac{1}{k_0}\sqrt{\frac{T}{b}}x+1\right)^2 \left(\Delta + Tx^2\right)e^{-x^2}dx$$

We are interested in low temperature and short mean interparticle separation: $\frac{T}{k_0^2 b} << 1$ The contributing parts of the integrand are symmetric, by that the energy takes the form:

$$E(T) = \frac{Vk_0^2}{2\pi^2}\sqrt{\frac{T}{b}}e^{-\frac{\Delta}{T}}\int_0^\infty \left(\Delta + Tx^2\right)e^{-x^2}dx$$

$$E(T) = \frac{Vk_0^2}{2\pi^2}\sqrt{\frac{T}{b}}\left(\Delta\frac{\sqrt{\pi}}{2} + T\frac{\sqrt{\pi}}{4}\right)e^{-\frac{\Delta}{T}}$$

$$E_{rotons} = \frac{Vk_0^2}{4\pi^{\frac{3}{2}}\sqrt{b}}\left(\Delta\sqrt{T} + \frac{T^{\frac{3}{2}}}{2}\right)e^{-\frac{\Delta}{T}}$$

Thus the heat capacity is:

$$\frac{\partial}{\partial T}\left(E_{phonos} + E_{rotons}\right) = \frac{2V\pi^2}{15c^3}T^3 + \frac{Vk_0^2}{4\pi^{\frac{3}{2}}\sqrt{b}}\left(\frac{3}{4}\sqrt{T} + \frac{\Delta}{\sqrt{T}} + \frac{\Delta^2}{T^{\frac{3}{2}}}\right)e^{-\frac{\Delta}{T}}$$

And

$$A(T) = \frac{2V\pi^2}{15c^3}T^3$$

$$B(T) = \frac{Vk_0^2}{4\pi^{\frac{3}{2}}}\left(\frac{3}{4}\sqrt{T} + \frac{\Delta}{\sqrt{T}} + \frac{\Delta^2}{T^{\frac{3}{2}}}\right)$$

(b) For two dimentional phonons system:

$$N(\omega)_{phonons} = \frac{A}{2\pi c^2}\int_0^\infty \omega d\omega$$

$$E(T)_{phonons} = \frac{AT^3}{2\pi c^2}\int_0^\infty \frac{1}{e^{\frac{\omega}{T}}-1}\left(\frac{\omega}{T}\right)^2 d\left(\frac{\omega}{T}\right) = \frac{AT^3}{2c^2\sqrt{\pi}}$$

For two dimentional rotons system:

$$E(k)_{rotons} = \frac{A}{2\pi c^2}\int_0^\infty e^{-\frac{\Delta + b(|\boldsymbol{k}|-k_0)^2}{T}}(\Delta + b(|\boldsymbol{k}|-k_0)^2)kdk$$

$$E(T)_{rotons} = \frac{Ak_0}{2\pi c^2}\sqrt{\frac{T}{b}}e^{-\frac{\Delta}{T}}\int_0^\infty \left(\frac{1}{k_0}\sqrt{\frac{T}{b}}x+1\right)\left(\Delta + Tx^2\right)e^{-x^2} = \frac{Ak_0}{4c^2}\sqrt{\frac{T}{b}}\left(\Delta\sqrt{\pi} + T\frac{\sqrt{\pi}}{2}\right)e^{-\frac{\Delta}{T}}$$

Without further assumpstion, we can deduce: $A(T)_{2D} \propto \frac{1}{T}A(T)_{3D}$ and $B(T)_{2D} \propto B(T)_{3D}$