Ex3042: Oscillations of a piston in a cylinder filled with gas

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The problem:

Consider a vertically aligned cylinder whose basis has an area A. A piston that has mass M is pushed from above. The piston is held by a spring that has an elastic constant K. If the cylinder is empty the piston is down at zero height (x = 0). The cylinder is filled with N gas particles. Each particle has mass m and the temperature is T. Consequently the the piston goes up a distance x, such that the gas occupies a volume Ax. Consider the following 3 cases:

(a) The temperature is high, such that Boltzmann approximation can be applied.

(b) The particles are condensed Bosons, T is lower than the condensation temperature.

(c) The particle are spinless Fermions, and the temperature is zero.

Answer the following questions, relating to each case separately.

1. What is the equilibrium position x_{eq} of the piston?
2. What is the frequency ω of small oscillations?
3. Plot schematic drawing of ω versus T.

Express answers using A, M, K, N, T. The schematic drawing is required to be be clearly displayed.

The solution:

In the equilibrium position x_{eq} the net force on the piston has to be 0. The force applied via the Spring is $-Kx_{eq}$ and the force applied via the gas filling the cylinder is PA, giving us

$$(1)\quad PA = Kx_{eq}$$

for all 3 cases.

(a) For a high temperature Boltzman gas, the equation of state is described by the law of ideal gasses $PV = NK_BT$ where in our case V=Ax_{eq}, giving us

$$(2)\quad PAx_{eq} = NK_BT$$

Substituting (1) for PA we get

$$(3)\quad Kx_{eq}^2 = NK_BT$$

$$(4)\quad x_{eq} = \sqrt{\frac{NK_BT}{K}}$$

(b) For bosons in the condensed state, the pressure depends only on the temperature and is given by

$$(5)\quad P = \frac{K_BTg_{\frac{5}{2}}(1)}{\lambda_T^3}$$

Where

$$(6)\quad g_{\frac{5}{2}}(1) = 1.342$$

and

$$(7)\quad \lambda_T = \sqrt{\frac{2\pi\hbar^2}{mK_BT}}$$

Giving us with (1):

$$(8)\quad x_{eq} = \frac{1.342K_BA}{\sqrt{\frac{2\pi\hbar^2}{mK_B}}^3}T^{-\frac{1}{2}}$$

(c) For Fermions without regard for spin under 0 temperature, the particles will fill the lowest N states in the system the energy of a given state is

$$(9)\quad E = \frac{\hbar^2\vec{k}^2}{2m}$$

Where

$$(10)\quad \vec{k} = \frac{\vec{n}\pi}{V^{\frac{1}{3}}}$$

and n is a non-negative whole number vector. The maximal value of n is given by the condition on the number of particles

$$(11)\quad N = \frac{\pi n_{max}^3}{6}$$

$$(12)\quad n_{max} = (\frac{6N}{\pi})^{\frac{1}{3}}$$

Hence by integration over all occupied states we get:

$$(13)\quad E_{total} = \int_0^{n_{max}} \frac{\pi n^2}{2}\frac{\hbar^2\pi^2 n^2}{2mV^{\frac{1}{3}}}dn = \frac{\hbar^2\pi^3}{4mV^{\frac{1}{3}}}\frac{n_{max}^5}{5}$$

Substituting (12) we get

$$(14)\quad E_{total} = \frac{\hbar^2\pi^3}{20m}\left(\frac{6N}{\pi}\right)^{\frac{5}{3}}V^{-\frac{2}{3}}$$

And since

$$(15)\quad P = -\frac{dE}{dV}$$

We get

$$(16)\quad P = \frac{\hbar^2\pi^3}{30m}\left(\frac{6N}{\pi}\right)^{\frac{5}{3}}V^{-\frac{5}{3}} = \frac{\hbar^2\pi^3}{30m}\left(\frac{6N}{\pi}\right)^{\frac{5}{3}}(Ax_{eq})^{-\frac{5}{3}}$$

Using (1) we get

$$(17)\quad \frac{K}{A}x_{eq} = \frac{\hbar^2\pi^3}{30m}\left(\frac{6N}{\pi}\right)^{\frac{5}{3}}(Ax_{eq})^{-\frac{5}{3}}$$

And so we have

$$(18)\quad x_{eq} = \left(\frac{\hbar^2\pi^3}{30m}\right)^{\frac{3}{8}}\left(\frac{6N}{\pi}\right)^{\frac{5}{8}}A^{-\frac{1}{4}}K^{\frac{3}{8}}$$

2) In order to find the frequency of the oscillations, we need to calculate the net force on the cylinder after a perturbation Δx, $\Delta F = -K_{total}\Delta x = -(K_{spring} + K_{gas})\Delta x$. The force applied by the gas

is $F_{gas} = PA$, and the difference in the force applied is $\Delta F_{gas} = \Delta PA$. Given that the mass of the cylinder is M, we get that the frequency of small oscilations is (from Classical Mechanics):

$$(19) \quad \omega \ = \sqrt{\frac{K+K_{gas}}{M}}$$

with

$$(20) \quad K_{gas} \ = \Delta PA \ = \Delta x \frac{\partial P}{\partial x} A$$

In this discussion we will assume T stays constant (Isothermal compression). The Adiabatic process for the Boltzmann case will be discussed in the appendix.

(a) In the Boltzman case, pressure is related to the volume according to the ideal gas law:

$$(21) \quad PV \ = NK_B T$$

Seeing that $V = Ax$ we get

$$(22) \quad P \ = \frac{NK_B T}{Ax}$$

And so

$$(23) \quad \Delta P \ = -\frac{\Delta x N K_B T}{Ax^2} \ = -\frac{\Delta x}{x} P$$

Finally resulting in

$$(24) \quad \omega_{Boltzmann} \ = \sqrt{\frac{K+K}{M}} \ = \sqrt{\frac{2K}{M}}$$

b) For a boson gas condensate in a given temperature the pressure is independent of the volume, with a volume change resulting in change of the ground state's population. Since the volume is directly proportional to the pistons position x, a Δx change in the piston's position doesn't change the pressure of the gas. Hence ΔP =0 and K_{gas}=0, and as a result

$$(25) \quad \omega_{BEC} \ = \sqrt{\frac{K}{M}}$$

c) For the degenerate Fermionic state, P depends on the position of the piston like so:

$$(26) \quad P \ = \frac{\hbar^2 \pi^3}{30m} \left(\frac{6N}{\pi}\right)^{\frac{5}{3}} (Ax_{eq})^{-\frac{5}{3}}$$

Hence:

$$(27) \quad \Delta P \ = -\Delta x \frac{\hbar^2 \pi^3}{30m} \left(\frac{6N}{\pi}\right)^{\frac{5}{3}} (Ax_{eq})^{-\frac{8}{3}} \frac{5}{3} \ = -\frac{5}{3} P \frac{\Delta x}{x_{eq}} \ = -\frac{5}{3} \frac{K}{A} \Delta x$$

And we get

$$(28) \quad K_{gas} \ = \frac{5}{3} K$$

$$(29) \quad \omega_{Fermions} \ = \sqrt{\frac{8K}{3M}}$$

3) At sufficiently high temperatures both fermionic and bosonic cases approach the Boltzman case.

For the bosonic case, so long that $T < T_c$, $\omega = \sqrt{\frac{K}{M}}$. Overall we get the following schematic:

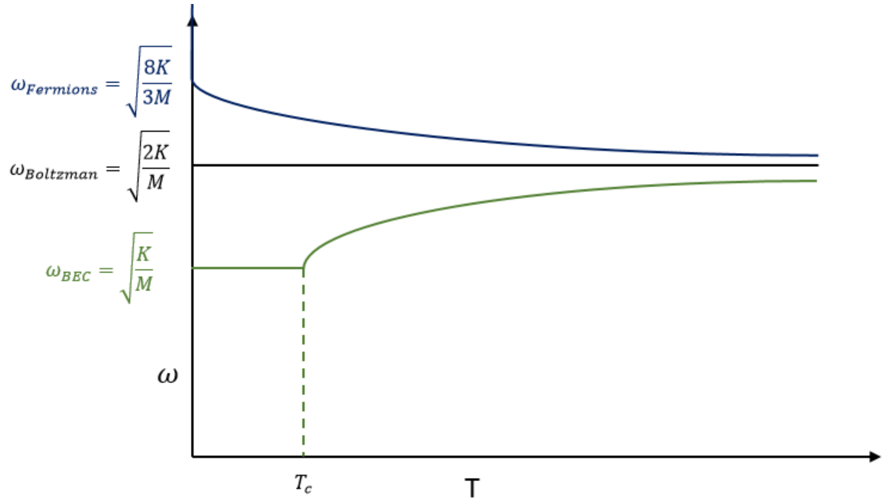

Figure 1: Schematic drawing of the frequency ω as a function of the temperature T for a piston with mass M suspended by a spring with spring constant K, pressurising Fermions (blue), classical Boltzman gas (black) and bosons (green)

Appendix In the adiabatic case of the Boltzman gas aproximation, ΔP is influenced by the temperature change of the gas in the cylinder. Deriving the temperature change via the connection $E = \frac{3}{2}NK_BT$ for ideal monoatomic gas. We can derive that the temperature component of ΔP is $-\frac{2}{3}P\frac{\Delta x}{x_{eq}}$, giving us the total K_{gas} of $\frac{5}{3}K$ and the frequency is

$$\omega_{adiabatic} = \sqrt{\frac{8K}{3M}}$$

Interestingly, this is the same result as the zero temperature fermionic case.