E3040: Quantum Bose Gas with an oscillating piston

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The problem:

A cylinder of length L and cross section A is divided into two compartments by a piston. The piston has mass M and it is free to move without friction. Its distance from the left basis of the cylinder is denoted by x. In the left side of the piston there is an ideal Bose gas of N_a particles with mass m_a. In the right side of the piston there is an ideal Bose gas of N_b particles with mass m_b. The temperature of the system is T.

(*) Assume that the left gas can be treated within the framework of the Boltzmann approximation.
(**) Assume that the right gas is in condensation.

(1)Find the equilibrium position of the piston.
(2)What is the condition for (*) to be valid?
(3)Below which temperature (**) holds?
(4)What is the frequency of small oscillations of the piston.

The solution:

(1) In equilibrium $P_A = P_B$.

For an Ideal gas $P_A A x = N_A T$.

For a Bose gas in condensation $P_B = \zeta\left(\frac{5}{2}\right)\left(\frac{m_b}{2\pi}\right)^{\frac{3}{2}} T^{\frac{5}{2}}$.

Let us define x_0 as the position of the piston at equilibrium.

Using the fact that in equilibrium $P_A = P_B$ we get $x_0 = \frac{N_A T^{\frac{-3}{2}}}{A\zeta\left(\frac{5}{2}\right)\left(\frac{m_b}{2\pi}\right)^{\frac{3}{2}}}$

(2) In order to use the Boltzmann approximation we demand low density in section A:

$\Rightarrow N_A \lambda_T^3 \ll V = A x_0$

$\frac{1}{\sqrt{m_a T}} \ll \left(\frac{A x_0}{N_A}\right)^{\frac{1}{3}}$

$\Rightarrow m_b \ll m_a$

(3)The total number of particles in side B is $N_b = N_b(\text{ground state}) + N_b(\text{excited states})$. We want

$N_b > N_b(\text{excited states})$

$N_b > \int_0^\infty g(\epsilon) f(\epsilon)\, \mathrm{d}\epsilon$.

Where $f(\epsilon, \mu \to 0) = \frac{1}{e^{\beta(\epsilon-\mu)}-1}$

And $g(\epsilon) = \frac{1}{4\pi^2}(2m_b)^{\frac{3}{2}} A(L - x_0)\sqrt{\epsilon}$

$\Rightarrow N_b > A(L - x_0)\zeta\left(\frac{3}{2}\right)\left(\frac{m_b}{2\pi}\right)^{\frac{3}{2}} T^{\frac{3}{2}}$

$\Rightarrow T^{\frac{3}{2}} < \left(\frac{m_b}{2\pi}\right)^{\frac{3}{2}} \left(\frac{N_b}{\zeta\left(\frac{3}{2}\right)} + \frac{N_a}{\zeta\left(\frac{5}{2}\right)}\right) \frac{1}{AL}$

(4)For small oscillations around the equilibrium position we get:

$\frac{N_a T}{x_0+dx} - \frac{N_a T}{x_0} \approx \frac{-N_a T dx}{x_0^2} = -Mw^2 dx$

$\Rightarrow w^2 = \frac{N_a T}{M x_0^2}$

$w = \zeta\left(\frac{5}{2}\right) \frac{A}{\sqrt{N_a M}} \left(\frac{m_b}{2\pi}\right)^{\frac{3}{2}} T^2$