Ex3030: Charged Bose gas in a divided box

Submitted by: Or Dobkowski

The problem:

Consider N bosons with mass m, positive charge e, and spin 0. The particles are in a box that is divided into two regions: zero voltage region of volume Ω_0, and voltage V region of volume Ω_v. Assume that the bosons are condensed in the Ω_0 region. In items (3,5) assume that the gas in the Ω_v region can be treated using the Boltzmann approximation.

(1) Find the $V = \infty$ condensation temperature $T_c(\infty)$.

(2) Find the $V = 0$ condensation temperature $T_c(0)$.

(3) Assuming an intermediate temperature, find the critical voltage V_c below which the bosons are no longer condensed.

(4) Write an exact expression for the energy $E(T, V)$ of the system

(5) Write an expression for the heat capacity $C(T, V)$ of the system. Keep only the leading correction in V.

Express the results using the thermal wavelength λ_T, the variables $\Omega_0, \Omega_v, N, T, eV$, and the functions $L_\alpha(z)$ and $\zeta(\alpha)$.

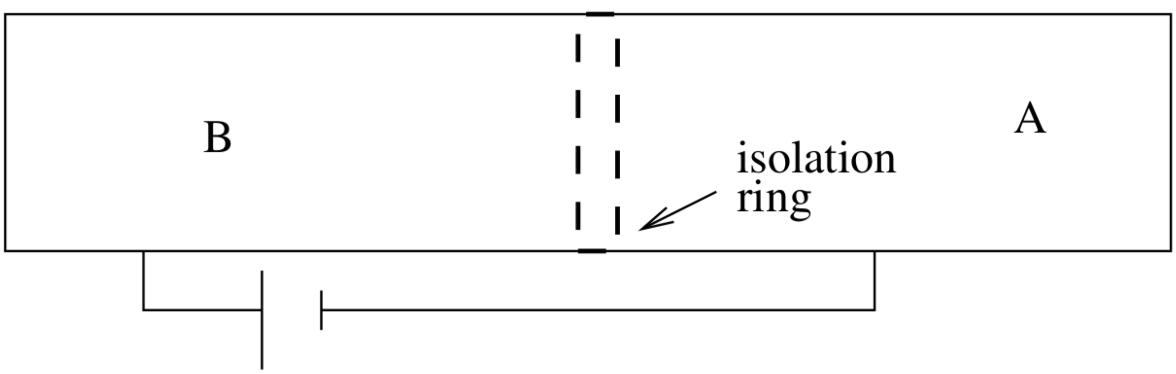

Figure 1: fig1

In (1) & (4) we will use two known results for a spinless non-relativistic bosons [given in the D.C lecture notes, and in K.Huang second edition p.289-290]

$$\frac{N}{\Omega} = \frac{1}{{\lambda_T}^3} Li_3/2(z) \tag{1}$$

$$\frac{E}{\Omega} = \frac{3}{2}\frac{T}{{\lambda_T}^3} Li_5/2(z) \tag{2}$$

(Ω is the sign for volume in this exercise)

The solution:

(1) $V = \infty$ means there will be no particles in the high voltage part of the box B. In that case we can consider just the Ω_0 part. In this volume the chemical potential is $\mu = 0 \rightarrow z = 1$, we use eq.(1) and solve for N

$$N = n_0 + \Omega\zeta\left(\frac{3}{2}\right)\left(\frac{m}{2\pi}\right)^{\frac{3}{2}} T^{\frac{3}{2}} \tag{3}$$

to get the critical temperature one simply sets $n_0 = 0$ (meaning zero particles in the condensation), set $\Omega = \Omega_0$ and solve for T_c

$$T_c = \left(\frac{2\pi}{m}\right)\left(\frac{N}{\Omega_0}\frac{1}{\zeta\left(\frac{3}{2}\right)}\right)^{\frac{2}{3}} \tag{4}$$

(2) For $V = 0$ we just have one box, with volume $\Omega_0 + \Omega_v$

$$T_c = \left(\frac{2\pi}{m}\right)\left(\frac{N}{\Omega_0 + \Omega_v}\frac{1}{\zeta\left(\frac{3}{2}\right)}\right)^{\frac{2}{3}} \tag{5}$$

(3) Assuming $T_c(0) < T < T_c(\infty)$, the number of particles in region Ω_0 is given by eq.(3) For the Ω_v region assuming the density is low, the Boltzmann approximation gives $\frac{N}{\Omega} = \frac{1}{{\lambda_T}^3} e^{\frac{-\epsilon}{T}}$, so that

$$N = n_0 + \Omega_0\zeta\left(\frac{3}{2}\right)\frac{1}{{\lambda_T}^3} + \Omega_v\frac{1}{{\lambda_T}^3} e^{\frac{-eV}{T}} \tag{6}$$

setting $n_0 = 0$ and solving for eV_c one gets

$$eV_c = T\ln\frac{\Omega_v}{N{\lambda_T}^3 - \Omega_0\zeta\left(\frac{3}{2}\right)} \tag{7}$$

(4) To find the exact energy of the system one should go back to the basics, and start from the integral

$$E = \int_0^\infty g(\epsilon)\epsilon f(\epsilon - \mu)d\epsilon \tag{8}$$

where $g(\epsilon)$ is the density of states, and $f(\epsilon - \mu)$ is the Bose occupation function.

$$g\left(\epsilon_k\right) = \Omega\frac{2m^{3/2}}{(2\pi)^2}\epsilon_k^{1/2} \tag{9}$$

$$f(\epsilon-\mu)=\frac{1}{e^{\beta(\epsilon-\mu)}-1} \tag{10}$$

We have two regions to treat: In both regions $\mu = 0$ since we have chemical equilibrium, and the chemical potential the a condensation is zero. In the Ω_0 region, $\epsilon = \epsilon_k$, so we get $E_0 = \int_0^\infty g(\epsilon_k)\epsilon_k f(\epsilon_k)d\epsilon_k$ which leads to eq(2) with $z = 1$, thus we get

$$E_0 = \Omega\frac{3}{2}\frac{T}{{\lambda_T}^3}Li_{5/2}(1) = \Omega_0\frac{3}{2}\frac{T}{{\lambda_T}^3}\zeta\left(\frac{5}{2}\right) \tag{11}$$

In the Ω_v region, $\epsilon = \epsilon_k + eV$, so $g\left(\epsilon\right) = g\left(\epsilon_k + eV\right) = g\left(\epsilon_k\right)$ and $d\epsilon = d(\epsilon_k + eV) = d\epsilon_k$

$$E_v = \int_0^\infty g(\epsilon_k)(\epsilon_k+eV)f(\epsilon_k+eV)d\epsilon_k = \int_0^\infty g(\epsilon_k)\epsilon_k f(\epsilon_k+eV)d\epsilon_k + \int_0^\infty g(\epsilon_k)eVf(\epsilon_k+eV)d\epsilon_k \tag{12}$$

The first integral is the same as eq (2) with $z = e^{\frac{-eV}{T}}$, the second integral is the same as eq(1) times a constant eV and $z = e^{\frac{-eV}{T}}$ thus we get

$$E_v = \Omega_v\left(\frac{3}{2}\frac{T}{{\lambda_T}^3}Li_{(5/2)}\left(e^{\frac{-eV}{T}}\right) + eV\frac{1}{{\lambda_T}^3}Li_{(3/2)}\left(e^{\frac{-eV}{T}}\right)\right) \tag{13}$$

$$E_{total} = \Omega_0\frac{3}{2}\frac{T}{{\lambda_T}^3}\zeta\left(\frac{5}{2}\right) + \Omega_v\left(\frac{3}{2}\frac{T}{{\lambda_T}^3}Li_{(5/2)}\left(e^{\frac{-eV}{T}}\right) + eV\frac{1}{{\lambda_T}^3}Li_{(3/2)}\left(e^{\frac{-eV}{T}}\right)\right) \tag{14}$$

(5) In the Boltzmann approximation $Li(z) \approx z$

$$E_{total} \approx \Omega_0\frac{3}{2}\frac{T}{{\lambda_T}^3}\zeta\left(\frac{5}{2}\right) + \Omega_v\left(\frac{3}{2}\frac{T}{{\lambda_T}^3}e^{\frac{-eV}{T}} + eV\frac{1}{{\lambda_T}^3}e^{\frac{-eV}{T}}\right) \tag{15}$$

$$\frac{1}{{\lambda_T}^3} = \left(\frac{mT}{2\pi}\right)^{3/2} \tag{16}$$

$$C = \frac{\partial E_{total}}{\partial T} \tag{17}$$

Differentiating and keeping the leading order in $\frac{eV}{T}$ we get

$$C \approx \Omega_0\frac{15}{4}\frac{1}{{\lambda_T}^3}\zeta\left(\frac{5}{2}\right) + \frac{\Omega_v}{{\lambda_T}^3}\left(\frac{eV}{T}\right)^2 e^{\frac{-eV}{T}} \tag{18}$$