# **Ex3220:** Bosons with spin in harmonic trap

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**The problem:**
$N$ Bosons that have spin1 are placed in a 3D harmonic trap. The harmonic trap frequency is $\Omega$. A magnetic field $B$ is applied, such that the interaction is $-\gamma B S_z$, where $S_z = 1, 0, -1$, and $\gamma$ is the gyromagnetic ratio.

(1) Write an expression for the density of one-particle states $g(\epsilon)$.

(2) Write an expression for the $B = \infty$ condensation temperature $T_c$.

(3) Write an equation for $T_c(B)$. It should be expressed in terms of the appropriate polylogarithmic funtion.

(4) Find the leading correction in $T_c(B)/T_c \approx 1 + \cdots$ assuming that $B$ is very large. It should be expressed in terms of an elementary function.

(5) Find what is $T_c(B)/T_c$ for $B = 0$, and what is the first-order correction term if $B$ is very small.

(6) Sketch a schematic plot of $T_c(B)/T_c$ versus $B$. Indicate by solid line the exact dependence, and by dashed and dotted lines the approximations. It should be clear from the figure whether the approximation under-estimates or over-estimates the true result, and what is the $B$ dependence of the slope.

**Tips:** The prefactors are important in this question. Do not use numerical substitutions.
Use the notation $L_\alpha(z)$ for the polylogarithmic function, and recall that $L_\alpha(1) = \zeta(\alpha)$.
Note also that $L'_\alpha(z) = (1/z)L_{\alpha-1}(z)$, and that $\Gamma(n) = (n-1)!$ for integer $n$.

**The solution:**

(1) For a 3D harmonic trap the energy is $E_n = (n_x + n_y + n_z)\Omega$. For a given enegy $\epsilon$ possible states should fulfill $n_x + n_y + n_z \leq \epsilon/\Omega$. The total amount of states possible:

$$N(\epsilon) = \int_0^{\epsilon/\Omega} \mathrm{d}n_x \int_0^{\epsilon/\Omega - n_x} \mathrm{d}n_y \int_0^{\epsilon/\Omega - n_x - n_y} \mathrm{d}n_z = \frac{1}{6}\left(\frac{\epsilon}{\Omega}\right)^3 \tag{1}$$

$$g(\epsilon) = \frac{\mathrm{d}N(\epsilon)}{\mathrm{d}\epsilon} = \frac{1}{2}\frac{\epsilon^2}{\Omega^3} \tag{2}$$

(2) The known result for $S_z = 0$:

$$N = cT^\alpha F_\alpha(z) \tag{3}$$

In our case: $c = \frac{1}{2\Omega^3}$ , $\alpha = 3$ , $F_3(z) = \Gamma(3)\mathrm{L}_3(z) = 2\mathrm{L}_3(z)$, so (3) becomes:

$$N = \left(\frac{T}{\Omega}\right)^3 \mathrm{L}_3(z) \tag{4}$$

And the total amount of particles:

$$N = \left(\frac{T}{\Omega}\right)^3 [\mathrm{L}_3(z\mathrm{e}^{\beta\gamma B}) + \mathrm{L}_3(z) + \mathrm{L}_3(z\mathrm{e}^{-\beta\gamma B})] \tag{5}$$

Below condensation temprature $ze^{\beta\gamma B} = 1 \implies z = e^{-\beta\gamma B}$. so (5) becomes:

$$N = (\frac{T}{\Omega})^3[L_3(1) + L_3(e^{-\beta\gamma B}) + L_3(e^{-2\beta\gamma B})] + \langle n_0 \rangle \tag{6}$$

For $B = \infty$ at $T = T_c$:

$$N = \left(\frac{T_c}{\Omega}\right)^3 L_3(1) \implies T_c = \Omega\left(\frac{N}{\zeta(3)}\right)^{1/3} \tag{7}$$

(3) The transcendental equation for $T_c(B)$:

$$N = (\frac{T}{\Omega})^3[\zeta(3) + L_3(e^{-\gamma B/T}) + L_3(e^{-2\gamma B/T})] \tag{8}$$

(4) We want to solve (8) for $B \gg 1$ in leading order for $x = (T/T_c) - 1$ where $T_c$ is the zero order solution. Using $L_\alpha(z \ll 1) \approx z$ we get:

$$L_3(e^{-\gamma B/T}) + L_3(e^{-2\gamma B/T}) \approx e^{-\gamma B/T} + e^{-2\gamma B/T} \tag{9}$$

$e^{-\gamma B/T}$ is the small parameter, so in order to simplify we will keep only first order terms (so we can neglect the last part in (9)). Equation (8) becomes:

$$\frac{T}{T_c} \approx \left(1 + \frac{e^{-\gamma B/T}}{\zeta(3)}\right)^{-1/3} \implies x \approx -\frac{1}{3\zeta(3)}e^{-\gamma B/T_c} \tag{10}$$

(5) For $B = 0$:

$$N = \left(\frac{T_c(0)}{\Omega}\right)^3 3L_3(1) \implies T_c(0) = \Omega\left(\frac{N}{3\zeta(3)}\right)^{1/3} = 3^{-1/3}T_c \tag{11}$$

For $B \ll 1$ we will use the approximation:

$$L_\alpha(z \approx 1) \approx L_\alpha(1) + \frac{(z-1)}{z}L'_\alpha(z) \approx \zeta(\alpha) + (z-1)\zeta(\alpha - 1) \tag{12}$$

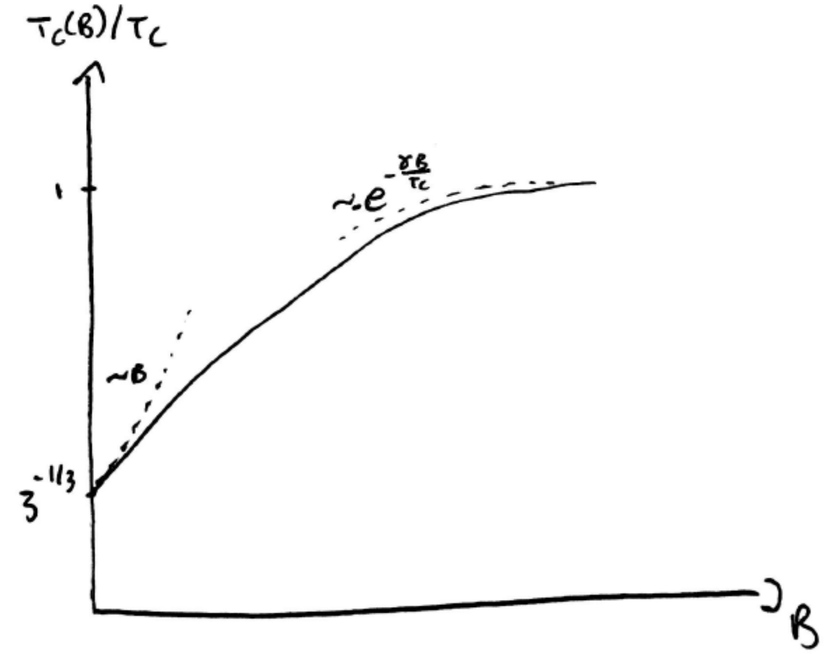
In order to simplify we take only first terms in $B$:

$$\begin{aligned} L_3(e^{-\gamma B/T}) + L_3(e^{-2\gamma B/T}) &\approx 2\zeta(3) + \zeta(2)[e^{-\gamma B/T} + e^{-2\gamma B/T} - 2] \\ &\approx 2\zeta(3) - \frac{3\gamma B}{T_c(0)}\zeta(2) \end{aligned} \tag{13}$$

And for the leading order in $x = T/T_c(0) - 1$ equation (8) becomes:

$$\frac{T}{T_c(0)} \approx \left(1 - \frac{\zeta(2)}{\zeta(3)}\frac{\gamma B}{T_c(0)}\right)^{-1/3} \implies x \approx \frac{1}{3}\frac{\zeta(2)}{\zeta(3)}\frac{\gamma B}{T_c(0)} \tag{14}$$

(6)




The large $B$ approximation cuts the $B = 0$ axis at $1 - \frac{1}{3\zeta(3)} \approx 0.723$, which is larger than $3^{-1/3} \approx 0.693$.