E3021: Bosons with Spin in magnetic field

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The problem:

N Bosons that have mass m and spin 1 are placed in a box that has volume V. A magnetic field B is applied, such that the interaction is $-\gamma B S_z$, where $S_z = 1, 0, -1$ and γ is the gyromagnetic ratio. In items (c-f) assume the Boltzmann approximation for the occupation of the $S_z \neq 1$ states.

(a) Find an equation for the condensation temperature T_c.

(b) Find the condensation temperature $T_c(B)$ for $B = 0$ and for $B \to \infty$.

(c) Find the critical B for condensation if T is set in the range of temperatures that has been defined in item(b).

(d) Describe how $T_c(B)$ depends of B in a qualitative manner. Find approximate expressions for moderate and large fields.

(e) Find the condensate fraction as a function of T and B.

(f) Find the heat capacity of the gas assuming large but finite field.

The solution:

(a) First, we can see that the Hamiltonian is $\mathcal{H} = \frac{p^2}{2m} - \gamma B S_z$, where $S_z = -1, 0, 1$. Therefore, we can treat the particles of the system like three different gasses.

From the lecture we know that

$$N = \sum_r f(\epsilon_r - \mu)$$

So for our case we get:

$$N = \underbrace{\sum_p \frac{1}{e^{\frac{\beta p^2}{2m} - \beta\gamma B - \beta\mu} - 1}}_{S_z=+1} + \underbrace{\sum_p \frac{1}{e^{\frac{\beta p^2}{2m} - \beta\mu} - 1}}_{S_z=0} + \underbrace{\sum_p \frac{1}{e^{\frac{\beta p^2}{2m} + \beta\gamma B - \beta\mu} - 1}}_{S_z=-1}$$

So n equals to

$$n = \frac{N}{V} = \frac{1}{\lambda_T^3}\left[Li_{3/2}(e^{\beta\gamma B + \beta\mu}) + Li_{3/2}(e^{\beta\mu}) + Li_{3/2}(e^{-\beta\gamma B + \beta\mu})\right]$$

The lowest energy state is for $S_z = +1$ particles. At the condensation we get

$$e^{\beta\gamma B + \beta\mu} = 1 \Rightarrow e^{\beta\mu} = e^{-\beta\gamma B} \Rightarrow \mu = -\gamma B.$$

At $T < T_c$

$$n = \frac{1}{\lambda_T^3}\left[Li_{3/2}(1) + Li_{3/2}(e^{-\beta\gamma B}) + Li_{3/2}(e^{-2\beta\gamma B})\right] + < n_0 >$$

(b) For $B = 0$, n is

$$n = \frac{3}{\lambda_T^3} Li_{3/2}(1) = 3 Li_{3/2}(1) \cdot \left(\frac{m T_c}{2\pi}\right)^{3/2}$$

So the condensation temperature is

$$T_c = \left(\frac{n}{3 \cdot 2.612}\right)^{2/3} \cdot \frac{2\pi}{m}$$

*Note that $Li_{3/2}(1) \approx 2.612$.

For $B \to \infty$: We have $\gamma B \gg T$, so the terms containing $e^{-2\beta\gamma B}$ are going to zero.

Hence, we get

$$n \approx \frac{1}{\lambda_T^3} Li_{3/2}(1)$$

So the condensation temperature is

$$T_c \approx \left(\frac{n}{2.612}\right)^{2/3} \cdot \frac{2\pi}{m}$$

(c+d) For the next items we can assume the Boltzmann approximation for the cases of $S_z \neq 1$: $\gamma B \gg T$.

If we calculate the ratio between the condensation temperature $T_c(B)$ for $B = 0$ and for $B \to \infty$, we get the following number

$$\frac{T_c^{B=0}}{T_c^{B\to\infty}} \approx \frac{1}{3^{2/3}}$$

As B is increased, T_c rises until B_c is reached. At $B = B_c$, $T = T_c$ and the condensation occurs. Then we have from Boltzmann

$$n\lambda_T^3 = Li_{3/2}(1) + Li3/2(e^{-\beta\gamma B_c}) \approx Li_{3/2}(1) + e^{-\beta\gamma B_c}$$

Notice we have neglected the third term in n, because it is of the second order in $e^{\beta\gamma B}$ and we only need the first.

Hence the critical magnetic field is

$$B_c = \frac{-T}{\gamma} ln\left(n\lambda_T^3 - Li_{3/2}(1)\right) \approx \frac{-T}{\gamma} ln\left(n\lambda_T^3 - 2.612\right)$$

(e) We need to calculate $\frac{<n_0>}{n}$. We have already showed that

$$n = \frac{1}{\lambda_T^3}\left(Li_{3/2}(1) + e^{-\beta\gamma B}\right) + < n_0 >$$

So the condensate fraction as function of T and B is

$$\frac{< n_0 >}{n} = 1 - \frac{Li_{3/2}(1) + e^{-\beta\gamma B}}{n\lambda_T^3}$$

(f) In order to calculate the heat capacity of the gas, we first need to write an expression for the energy. Once again, we continue to look at the system as 3 different gasses, and again we can assume $\gamma B \gg T$. We get

$$\frac{E}{V} = \frac{3}{2} \cdot \frac{T}{\lambda_T^3}\left(Li_{5/2}(1) + e^{-\beta\gamma B}\right) = \frac{3}{2}\left(\frac{m}{2\pi}\right)^{3/2} \cdot T^{5/2}\left(Li_{5/2}(1) + e^{-\frac{\gamma B}{T}}\right)$$

Now, to get the heat capacity we differentiate with respect to T.

$$C_V = \frac{\partial}{\partial T}\left(\frac{E}{V}\right) = \frac{15}{4}\cdot\frac{1}{\lambda_T^3}\left(Li_{5/2}(1) + e^{-\beta\gamma B}\right) + \frac{3}{2}\cdot\frac{1}{\lambda^3}\left(\frac{\gamma B}{T}\right)e^{-\beta\gamma B}$$

The third term is much more significant than the second term, therefore we can neglect the latter and get

$$C_V \approx \frac{15}{4}\cdot\frac{1}{\lambda_T^3}Li_{5/2}(1) + \frac{3}{2}\cdot\frac{1}{\lambda_T^3}\left(\frac{\gamma B}{T}\right)e^{-\beta\gamma B}$$