E3021: Bosons with Spin in magnetic field

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The problem:

N Bosons that have mass m and spin1 are placed in a box that has volume V. A magnetic field B is applied, such that the interaction is $-\gamma B S_z$, where $S_z = 1, 0, -1$ and γ is the gyromagnetic ratio. In items (c-f) assume the Boltzmann approximation for the occupation of the $S_z \neq 1$ states.

(a) Find an equation for the condensation temperature T_c.

(b) Find the condensation temperature $T_c(B)$ for $B = 0$ and for $B \to \infty$.

(c) Find the critical B for condensation if T is set in the range of temperatures that has been defined in item(b).

(d) Describe how $T_c(B)$ depends of B in a qualitatively manner. Find approximate expressions for moderate and large fields.

(e) Find the condensate fraction as a function of T and B.

(f) Find the heat capacity of the gas assuming large but finite field.

The solution:

(a) Before we find the condensation temperature T_c, we shold understand that we can treat the particles of the system like three different gasses, therfore the Hammiltonian is $\mathcal{H} = \frac{p^2}{2m} - \gamma B S_z$ where $s_z = -1, 0, 1$.

$$n = \frac{N}{V} = \underbrace{\sum_p \frac{1}{\frac{1}{\zeta}e^{\frac{\beta p^2}{2m} - \beta\gamma B} - 1}}_{s_z=+1} + \underbrace{\sum_p \frac{1}{\frac{1}{\zeta}e^{\frac{\beta p^2}{2m}} - 1}}_{s_z=0} + \underbrace{\sum_p \frac{1}{\frac{1}{\zeta}e^{\frac{\beta p^2}{2m} + \beta\gamma B} - 1}}_{s_z=-1}$$

So n equals to

$$n = \frac{1}{\lambda^3}[g_{3/2}(\zeta e^{\beta\gamma B}) + g_{3/2}(\zeta) + g_{3/2}(\zeta e^{-\beta\gamma B})]$$

The lowest energy state for $s_z = +1$ particles, all terms must be ≥ 0. At the condensation we get $\zeta e^{\beta\gamma B} = 1 \Rightarrow \zeta = e^{-\beta\gamma B}$. At $T < T_c$

$$n = \frac{1}{\lambda^3}[g_{3/2}(1) + g_{3/2}(e^{-\beta\gamma B}) + g_{3/2}(e^{-2\beta\gamma B})] + < n_0 >$$

(b) For $B = 0$, n is

$$n = \frac{3}{\lambda^3} g_{3/2}(1) = 3 g_{3/2}(1) \cdot (\frac{2m\pi k T_c}{h^2})^{3/2}$$

So the condensation temperature is

$$kT_c = \left(\frac{n}{3 \cdot 2.612}\right)^{2/3} \cdot \frac{2\pi\hbar^2}{m}$$

*Note that $g_{3/2}(1) \approx 2.612$.

The term $g_{3/2}(e^{-2\beta\gamma B})$ is equal to zero, if $\gamma B \gg kT$, because we keep only lowest order correction for large B.

For $\gamma B \gg kT$ n is

$$n = \frac{1}{\lambda^3}\left(g_{3/2}(1) + g_{3/2}(e^{-\beta\gamma B})\right)$$

Note that $g_v(z) = z + \frac{z^2}{2^v} + o(z^3)$. So for $z \ll 1$ we get $g_v(z) \approx z$. Hence,

$$n \approx \frac{1}{\lambda^3}\left(g_{3/2}(1) + e^{-\beta\gamma B}\right)$$

So the condensation temperature is

$$kT_c = \left(\frac{n}{g_{3/2}(1) + e^{-\beta\gamma B}}\right)^{2/3} \cdot \frac{2\pi\hbar^2}{m} \approx \left(\frac{n}{2.612}\right)^{2/3} \cdot \frac{2\pi\hbar^2}{m}$$

(c+d) If we calculate the ratio between the condensation temperature $T_c(B)$ for $B = 0$ and between for $B \to \infty$, we get the following number

$$\frac{kT_c^{B=0}}{kT_c^{B\to\infty}} \approx \frac{1}{3^{2/3}}$$

As B is increased, T_c rises until B_c is reached. At $B = B_c$, $T = T_c$ and the condensation occurs. Then we have from the case of $\gamma B \gg kT$

$$n\lambda^3 = g_{3/2}(1) + g_{3/2}(e^{-\beta\gamma B_c}) \approx g_{3/2}(1) + e^{-\beta\gamma B_c}$$

Hence the critical magnetic field is

$$B_c = \frac{-kT}{\gamma} ln\left(n\lambda^3 - g_{3/2}(1)\right)$$

(e) We should calculate $\frac{<n_0>}{n}$. We have already showed that in

$$n = \frac{1}{\lambda^3}\left(g_{3/2}(1) + e^{-\beta\gamma B}\right) + <n_0>$$

So the condensate fraction as function of T and B is

$$\frac{<n_0>}{n} = 1 - \frac{g_{3/2}(1) + e^{-\beta\gamma B}}{n\lambda^3}$$

(f) Before we can calculate the heat capacity of the gas, we have to formulate an equation for the energy.

$$E = -\left(\frac{\partial}{\partial\beta} ln(Z)\right) = VkT^3 \sum_{s_z=-1,0,1} \frac{\partial}{\partial T}\left(\frac{g_{3/2}(\zeta e^{\beta\gamma B S_z})}{\lambda^3}\right)_{\zeta,V}$$

At $T \leq T_c$:

$$\frac{E}{V} = kT^2 \frac{\partial}{\partial T}\left[\frac{g_{3/2}(1)}{\lambda^3} + \frac{e^{-\beta\gamma B}}{\lambda^3} + \frac{e^{-2\beta\gamma B}}{\lambda^3}\right]$$

$$\approx kT^2 \frac{\partial}{\partial T}\left[\frac{g_{3/2}(1)}{\lambda^3} + \frac{e^{-\beta\gamma B}}{\lambda^3}\right] = \frac{kT}{\lambda^3}\cdot\frac{3}{2}\left(g_{3/2}(1) + e^{-\beta\gamma B}\right) + \frac{kT}{\lambda^3}\cdot\frac{\gamma B}{kT}e^{-\beta\gamma B}$$

$$\approx \frac{kT}{\lambda^3}\cdot\frac{3}{2}g_{3/2}(1) + \frac{\gamma B}{\lambda^3}e^{-\beta\gamma B}$$

Note that for $\gamma B \gg kT$ the third term is more significant then the second one. Also, we have used the following results

$$\frac{\partial}{\partial T}\left(\frac{1}{\lambda^3}\right) = \frac{3}{2T}\cdot\frac{1}{\lambda^3}$$

$$\frac{\partial}{\partial T}\left(\frac{e^{-\beta\gamma B}}{\lambda^3}\right) = \frac{e^{-\beta\gamma B}}{\lambda^3}\cdot\left(\frac{3}{2T} + \frac{\gamma B}{kT^2}\right)$$

For heat capacity of the gas assuming large but finite field ($\gamma B \gg kT$) is

$$\frac{c_v}{k_B} = \frac{9}{4}\cdot\frac{1}{\lambda^3}g_{3/2}(1) + \frac{1}{\lambda^3}\left(\frac{\gamma B}{k_B T}\right)^2 e^{-\beta\gamma B}$$