E3010: Heat capacity of ideal Bose gas

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The problem:

N bosons with spin 1 and mass m are in a box with a volume V . The gas is in a thermic equilibrium in a temperature .

(1) Write an explicit expression for Tc using N, m and V.
(2) Calculate the chemical potential , the energy and the pressure in the Boltzmann approximation $T \gg Tc$.
(3) Calculate the chemical potential , the energy and the pressure in the regime $T < Tc$.
(4) calculate C_v in the temperature $T < T_c$
(5) What is the result for C_v in the temperatures T = Tc and in Boltzmann approximation $T \gg Tc$.
(6) Show that $\frac{C_p}{C_v} = \frac{5g_{\frac{5}{2}}(z)g_{\frac{1}{2}}(z)}{3g^2_{\frac{3}{2}}(z)}$.
Why is $C_p \to \infty$ in the condensed phase?.
(7) Find γ in the adiabatic equation of state. Note that in general $\gamma \neq \frac{C_p}{C_v}$.

The solution: Note an impoved version in [ex3009].

(1)$T = T_c$

$$T_c = \frac{1}{2m\pi}(\frac{N}{3V\zeta(\frac{3}{2})})^{\frac{2}{3}} = \frac{1}{2m\pi}(\frac{N}{3V2.612})^{\frac{2}{3}}. \tag{1}$$

(2) $T \gg T_c$ Boltzmann approximation.

$$\frac{N}{V} = 3\frac{z}{\lambda^3} \to z = \frac{N\lambda^3}{3V}. \tag{2}$$

$$E = 3\frac{3}{2}V\frac{T}{\lambda^3}z = \frac{9}{2}V\frac{T}{\lambda^3}e^{\beta\mu} = \frac{3}{2}NT. \tag{3}$$

$$P = \frac{2E}{3V} = 3\frac{T}{\lambda^3}e^{\beta\mu} = \frac{NT}{V}. \tag{4}$$

(3) $T < T_c$.

$$\mu = 0. \tag{5}$$

$$E = 3V(2m)^{\frac{3}{2}}2\pi\Gamma(\frac{5}{2})\zeta(\frac{5}{2})T^{\frac{5}{2}} = 3V(2m)^{\frac{3}{2}}2\pi\frac{3}{4}\sqrt{\pi}1.341T^{\frac{5}{2}}. \tag{6}$$

$$P=\frac{2E}{3V}=2(2m)^{\frac{3}{2}}2\pi\frac{3}{4}\sqrt{\pi}1.341T^{\frac{5}{2}}. \tag{7}$$

(4) $T<T_c$

$$C_v=\frac{\partial E}{\partial T}=3V(2m)^{\frac{3}{2}}2\pi\Gamma(\frac{5}{2})\zeta(\frac{5}{2})T^{\frac{3}{2}}\frac{5}{2}=3V(2m)^{\frac{3}{2}}2\pi\sqrt{\pi}1.341T^{\frac{3}{2}}\frac{15}{8}. \tag{8}$$

$$N_e=N\frac{T}{T_c}^{\frac{3}{2}} \tag{9}$$

particles in the excited states increased with T.

$C_v\propto N_e(T)$.
C_v increased with N_e until $N_e(T_c=T)=N$.

(5) $T=T_c$.

$$C_v=\frac{\partial E}{\partial T}=3V(2m)^{\frac{3}{2}}2\pi\Gamma(\frac{5}{2})\zeta(\frac{5}{2})T_c^{\frac{3}{2}}\frac{5}{2}=3V(2m)^{\frac{3}{2}}2\Pi\sqrt{\pi}1.341T_c^{\frac{3}{2}}\frac{15}{8}=1.925NK. \tag{10}$$

$C_v(T=T_c)$ Heat capacity have maximum, phase transition .

$T>T_c$.

$$C_v=\frac{\partial}{\partial T}[(T\frac{3g_{\frac{5}{2}}(z)}{2g_{\frac{3}{2}}(z)})]_v=\frac{15g_{\frac{5}{2}}}{4g_{\frac{3}{2}}(z)}-\frac{9g_{\frac{3}{2}}(z)}{4g_{\frac{1}{2}}(z)}. \tag{11}$$

for $T=T_c\rightarrow C_v=1.925NK$.

C_v grow smaller with T .

$T\gg T_c$. Boltzmann approximation.

$$E=\frac{3}{2}NT\rightarrow C_v=\frac{3}{2}N. \tag{12}$$

(6)

$$\frac{C_p}{NK}=\frac{\partial}{\partial T}[(T\frac{5g_{\frac{5}{2}}(z)}{2g_{\frac{3}{2}}(z)})]_p=[\frac{\partial z}{\partial T}]_p[\frac{\partial F(g)}{\partial z}]_p+F(g)=\frac{25}{4}\frac{g_{\frac{5}{2}}^2(z)g_{\frac{1}{2}}(z)}{g_{\frac{3}{2}}^3(z)}-\frac{15}{4}\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)}. \tag{13}$$

$$F(g)=\frac{5g_{\frac{5}{2}}(z)}{2g_{\frac{3}{2}}(z)}. \tag{14}$$

$$[\frac{\partial z}{\partial T}]_p = \frac{-5z}{2T}\frac{g_{\frac{5}{2}}(z)}{g_{\frac{3}{2}}(z)}. \tag{15}$$

C_v From (11)

$$\rightarrow \frac{C_p}{C_v} = \frac{5g_{\frac{5}{2}}(z)g_{\frac{1}{2}}(z)}{3g^2_{\frac{3}{2}}(z)}. \tag{16}$$

$C_p(T < T_c) \rightarrow \infty$ all particles in the condensed phase and they compress to zero volume the system have only one phase.

(7) Adiabat of an ideal Bose gas.

For $T \leq T_c$

S - Entropy

$$\frac{S}{NK} = \frac{5}{2}\frac{V}{\lambda^3}\zeta(\frac{5}{2}) \rightarrow VT^{\frac{3}{2}} = const. \tag{17}$$

$$P = \frac{KT}{\lambda^3}\zeta(\frac{5}{2}) \rightarrow \frac{P}{T^{\frac{5}{2}}} = const. \tag{18}$$

From (17) and (18)

$$PV^{\frac{5}{3}} = const \rightarrow \gamma = \frac{5}{3}. \tag{19}$$

Ideal Bose gas only for $T \gg T_c$ then $\gamma \simeq \frac{5}{3}$.