Ex2230: Harmonic oscillators, Photons

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The problem:

Find the state equations of photon gas in 1D/2D/3D cavity within the framework of the canonical formalism, regarding the electromagnetic modes as a collection of harmonic oscillators.

Note: additional exercises on photon gas and blackbody radiation can be found in the context of quantum gases. Formally, photon gas is like Bose gas with chemical potential $\mu = 0$. Note that the same type of calculation appears in Debye model (”acoustic” phonons instead of ”transverse” photons).

(1) Write the Partition function for a single photon.

(2) Find the occupation function for each ω mode.

(3) Find the degeneracy function $g(\omega)$.
Note: do not forget dependency on the dimension of the problem.

(4) Find the total Energy of a photon gas.

(5) Find the free Energy of a photon gas.

(6) Find an expression for the pressure of the photon gas.

The solution:

1 - Partition function

In this exercise we are supposed to treat a photon as a single harmonic oscillator ranging in frequencies.
The harmonic oscillator Hamiltonian is:

$$\hat{\mathcal{H}} = \hbar\omega \left(\hat{n} + \frac{1}{2}\right) \tag{1}$$

And we shall gauge the energy so the ground level energy contribution is zero:

$$\hat{\mathcal{H}} = \hbar\omega\hat{n} \tag{2}$$

Using the canonical ensemble we find the partition function for a single ”photon” to be:

$$Z_{\omega,1} = \sum_{n=0}^{\infty} \exp\left(-\beta\hbar\omega\hat{n}\right) = \left(1 - \exp\left(-\beta\hbar\omega\right)\right)^{-1} \tag{3}$$

2 - Occupation function

$$f(\omega) = <\hat{n}> = -\frac{1}{\hbar\omega}\frac{\partial\left(\ln(Z_{\omega,1})\right)}{\partial\beta} = \frac{1}{\exp\left(\beta\hbar\omega\right) - 1} \tag{4}$$

Notice this is the same treatment for Bose gas, with chemical potential $\mu = 0$, and the argument is by convention ω instead of ε.

3 - The degeneracy function $g(\omega)$

We want to count the number of k-modes possible for ω frequency:

$$\omega = \frac{k\pi c}{L} \tag{5}$$

Thus:

$$N = \int_{k=0}^{\frac{\omega L}{\pi c}} d^D k = \frac{\Omega^D}{2^D} \int k^{D-1} dk = \frac{\Omega^D k^D}{2^D \cdot D}\Bigg|_0^{\frac{\omega L}{\pi c}} = \frac{\Omega^D \left(\frac{L}{\pi c}\right)^D}{2^D \cdot D} \omega^D \tag{6}$$

Now

$$g(\omega) = \frac{\partial N}{\partial \omega} \Rightarrow g(\omega) = \begin{cases} \left(\frac{L}{\pi c}\right) & 1D \\ \frac{\pi}{2}\left(\frac{L}{\pi c}\right)^2 \omega & 2D \\ \frac{\pi}{2}\left(\frac{L}{\pi c}\right)^3 \omega^2 & 3D \end{cases} \tag{7}$$

4 - total energy of photon gas

(1D) In one dimension we have no degeneracy so the total energy is simply the number of oscillators (or photons) multiplied by the average oscillator energy:

$$E = N \cdot <\epsilon> = \frac{N\hbar\omega}{e^{\frac{\hbar\omega}{T}} - 1} \tag{8}$$

(2D)

$$E_{total} = P\int_0^\infty g(\omega)\hbar\omega f(\omega) d(\omega) = P \cdot \frac{\pi}{2}\left(\frac{L}{\pi c}\right)^2 \int_0^\infty \frac{\hbar\omega^2 \, d\omega}{e^{\frac{\hbar\omega}{T}} - 1} = \tag{9}$$

$$= P \cdot \frac{\pi}{2}\left(\frac{L}{\pi c}\right)^2 \cdot \hbar \left(\frac{T}{\hbar}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{P}{\pi}\zeta(3)\left(\frac{L}{\hbar c}\right)^2 T^3 \approx \frac{2.404}{\pi}\left(\frac{L}{\hbar c}\right)^2 T^3$$

Where P is the polarization number, and for photons it's 2
So all in all we have:

$$U = \frac{2.404}{\pi}\left(\frac{L}{\hbar c}\right)^2 T^3 \tag{10}$$

$$\rho(U) = \frac{2.404}{\pi}\left(\frac{1}{\hbar c}\right)^2 T^3 \tag{11}$$

Where here $\rho(U)$ is the energy density per unit area.

(3D)

$$E_{total} = P\int_0^\infty g(\omega)\hbar\omega f(\omega) = P\hbar \cdot \frac{\pi}{2}\left(\frac{L}{\pi c}\right)^3 \int_0^\infty \frac{\omega^3 \, d\omega}{e^{\frac{\hbar\omega}{T}} - 1} = \tag{12}$$

$$= P\hbar \cdot \frac{\pi}{2}\left(\frac{L}{\pi c}\right)^3 \frac{T^4}{\hbar^4} \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{P}{2}\frac{\pi^2}{15}\left(\frac{L}{\hbar c}\right)^3 T^4$$

Again with $P = 2$ we get:

$$U = \frac{\pi^2}{15}\left(\frac{L}{\hbar c}\right)^3 T^4 \tag{13}$$

$$\rho(U) = \frac{\pi^2}{15}\left(\frac{1}{\hbar c}\right)^3 T^4 \tag{14}$$

5 - Free Energy

We've already done most of the work needed to find the free energy function, as we've found $g(\omega)$, and the partition function for a single mode, so now, we just have to apply our previous knowledge:

(1D)

$$Z_w = \frac{1}{1-e^{-\hbar\omega\beta}} \tag{15}$$

$$Z_{all-modes} = \prod_{\omega} \frac{1}{1-e^{-\hbar\omega\beta}} \tag{16}$$

$$F \equiv -T\ln(Z) = T\sum_{\omega} \ln\left(1-e^{-\frac{\hbar\omega}{T}}\right) \to T\int_0^\infty g(\omega)\ln\left(1-e^{-\frac{\hbar\omega}{T}}\right)d\omega = \tag{17}$$

$$= \frac{LT}{\pi c}\left[\cancelto{0}{\omega\cdot\ln\left(1-e^{-\frac{\hbar\omega}{T}}\right)\Big|_0^\infty} + \frac{\hbar}{T}\int_0^\infty \frac{\omega\, d\omega}{e^{\frac{\hbar\omega}{T}}-1}\right] = \frac{LT^2}{\hbar\pi c}\int_0^\infty \frac{xdx}{e^x-1} = \frac{\pi}{6}\left(\frac{L}{\hbar c}\right)T^2$$

(2D)

$$F = T\sum_{\omega} \ln\left(1-e^{\frac{-\hbar\omega}{T}}\right) \to T\int_0^\infty g(\omega)\ln\left(1-e^{\frac{-\hbar\omega}{T}}\right)d\omega = \tag{18}$$

$$= T\frac{\pi}{2}\left(\frac{L}{\pi c}\right)^2\int_0^\infty \omega\ln\left(1-e^{\frac{-\hbar\omega}{T}}\right)d\omega = T\frac{\pi}{2}\left(\frac{L}{\pi c}\right)^2\frac{1}{2}\left[\cancelto{0}{\omega^2\cdot\ln\left(1-e^{\frac{-\hbar\omega}{T}}\right)\Big|_0^\infty} + \frac{\hbar}{T}\frac{\omega^2\, d\omega}{e^{\frac{\hbar\omega}{T}}-1}\right] =$$

$$= T\frac{\pi}{4}\left(\frac{L}{\pi c}\right)^2\left(\frac{T}{\hbar}\right)^2\int_0^\infty \frac{x^2dx}{e^x-1} = \frac{\zeta(3)}{2\pi}\left(\frac{L}{\hbar c}\right)^2 T^3$$

Taking polarization into account we get:

$$F = \frac{\zeta(3)}{\pi}\left(\frac{L}{\hbar c}\right)^2 T^3 \tag{19}$$

(3D)

$$F = ... = T\frac{\pi}{2}\left(\frac{L}{\pi c}\right)^3\int_0^\omega \omega^2\ln\left(1-e^{\frac{-\hbar\omega}{T}}\right)d\omega = \tag{20}$$

$$= T\frac{\pi}{6}\left(\frac{L}{\pi c}\right)^3\left[\cancelto{0}{\omega^3\ln\left(1-e^{\frac{-\hbar\omega}{T}}\right)\Big|_0^\infty} + \frac{\hbar}{T}\int_0^\infty \frac{\omega^3\, d\omega}{e^{\frac{\hbar\omega}{T}}-1}\right] = T\frac{\pi}{6}\left(\frac{L}{\pi c}\right)^3\left(\frac{T}{\hbar}\right)^3\int_0^\infty \frac{x^3dx}{e^x-1} =$$

$$= \frac{\pi^2}{90}\left(\frac{L}{\hbar c}\right)^3 T^4$$

Taking into account polarization number we get:

$$F = \frac{\pi^2}{45}\left(\frac{L}{\hbar c}\right)^3 T^4 \tag{21}$$

6 - State equation

Since we know the d-volume is the conjugate quantity of the d-pressure we can directly assign:

$$"P" = \frac{\partial F}{\partial "V"} \tag{22}$$

Where, for instance the 1 dimensional pressure is simply the force exerted by at the ends of the infinitesimal length unit (this is similar to tension, only operating outwards and not inwards)

(1D)

$$f = \frac{\partial F}{\partial L} = \frac{F}{L} = \frac{\pi}{6}\left(\frac{1}{\hbar c}\right)T^2 \tag{23}$$

(2D)

$$P_\lambda = \frac{\partial F}{\partial L^2} = \frac{F}{L^2} = F = \frac{\zeta(3)}{\pi}\left(\frac{1}{\hbar c}\right)^2 T^3 \tag{24}$$

(3D)

$$P = \frac{\partial F}{\partial V} = \frac{\partial F}{\partial L^3} = \frac{F}{L^3} = \frac{\pi^2}{45}\left(\frac{1}{\hbar c}\right)^3 T^4 = \frac{1}{3}\rho(U) \tag{25}$$

The last equation yields the known relation (in cosmology) between pressure and energy density

$$P = w \cdot \rho \tag{26}$$

where $\frac{1}{3}$ is indeed the relation for radiation.