Exercises in Statistical Mechanics

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This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

====== [Exercise 2130]

Photon gas as collection of harmonic oscillators

$$H=\frac{\hat{p}^2}{2m}+\frac{1}{2}mw^2\hat{x}^2$$

we define $\hat{n}=a^{+}a$, $a=\sqrt{\frac{w}{2}}\hat{x}-i\frac{1}{\sqrt{2w}}\hat{p}$, and then it's possible to write $H=w\hat{n}+const$. If $const=\frac{1}{2}w$, $\hat{n}$ is an operator with self states $|n>$ and the matching eigenvalues are $n=0,1,2....$ therefore, the stationary states of the oscillator are

$$|n>,\, E_n=nw$$

The next reinterpretation is acceptable for the states $|n>$: $|n>$ is a state where n particles ("bozons") occupies uniparticle level, with energy w. When the system, described as H is in equilibrium with the environment

$$P_n=\frac{1}{Z(\beta)}e^{-\beta E_n}=\frac{1}{Z(\beta)}e^{-\beta wn}$$

Prove $Z(\beta)=\frac{1}{1-e^{-\beta w}}$

$$\langle n\rangle=\frac{1}{e^{\beta w}-1}\equiv f(w)$$

$f(w)$ is called Bose's occupation function.
The hamiltonian that describes the electromagnetic field, if we use 'normal coordinates', is a collection of independent oscillators.

$$H=\sum_{k\alpha}w_{k\alpha}\hat{n_{k\alpha}}$$

We give the hamiltonian the next reinterpretation: $|k,\alpha>$ is a uniparticle state of photon with k momentum and α polarization. Periodic boundary conditions are $\vec{k}\rightarrow\frac{2\pi}{L}(m_1,m_2,m_3)$. Two transversal polarization directions $\alpha=1,2$. The photon energy in the state $|k,\alpha>$ is $w_{k\alpha}=|\vec{k}|$. The state $|...n_{k\alpha}...>$ is a multiparticle occupation state with $n_{k\alpha}$ photons, settelling the uniparticle state $|k\alpha>$. The occupation states $|...n_{k\alpha}...>$ are the stationary states of the system, and we get

$$E_{(...n_{k\alpha}...)}=\sum_{k\alpha}n_{k\alpha}w_{k\alpha}$$

In thermal equilibrium of the electromagnetic field, described with H, with the environment

$$P(...n_{k\alpha}...)=\frac{1}{Z}e^{-\beta E(...n_{k\alpha}...)}$$

(a) Calculate the average number of photons $N(T,V)$. The temperature and the volume of the medium are given.

(b) Calculate the distribution function $Z(\beta,V)$. Use factorization and integration in parts.

(c) Calculate the thermic energy of the field in two ways. show that you get

$$E\left(T,V\right)=\frac{\pi^2}{15}VT^4$$

(d) Find the radiation pressure. Is it possible to use the results from exercise 17 with $\nu=1$. Show that any way, Pressure$=\frac{1}{3}\left(\frac{E}{V}\right)$.

To solve paragraphs (a) - (d) use the next hints. always remember that photon gas = electromagnetic field = oscillator's collection, so there's no need to panic...
Hints to solve (a) - (d)

(a) The photon's number $N\left(T,V\right)=\sum_{k\alpha}\langle n_{k\alpha}\rangle$

$$\sum_{k\alpha}\rightarrow 2+S\left(\frac{L}{2\pi}\right)^2 d^2k,\,\langle n_{k\alpha}\rangle=f\left(w_{k\alpha}\right)$$

$$\int_0^\infty\frac{x^2dx}{e^x-1}=2\zeta\left(3\right)=2\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}...\right)$$

(b)

$$\ln Z\left(\beta\right)=\sum_{k\alpha}\ln Z^{(k,\alpha)}\left(\beta\right)=...$$

$$\int_0^\infty\frac{x^2dx}{e^x-1}=6\zeta\left(4\right)=\frac{\pi^{\ 4}}{15}$$

(c)

$$E\left(T,V\right)=\sum_{k\alpha}w_{k\alpha}\langle n_{k\alpha}\rangle=...$$

$$E\left(T,V\right)=-\frac{\partial\ln Z}{\partial\beta}=...$$

(e)

$$P\left(T,V\right)=\frac{1}{\beta}\frac{\partial\ln Z}{\partial V}=...$$

Additional materials about photon gas will be learned in the frame of the grand canonical ensemble. Formally, photon gas is like Bose gas with chemical potential $\mu=0$.