Exercises in Statistical Mechanics

Based on course by Doron Cohen, has to be proofed
Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel

This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

====== [Exercise 2090]

Electron gas in magnetic field

The interaction of the electron with the electromagnetic field is described by the hamiltonian ($e = -|e|$):

$$\hat{H} = \frac{1}{2m}\left(\hat{p} - e\hat{A}\right)^2 - rB\cdot\hat{\sigma}$$

For an homogenic magnetic field along axis Z it's possible to choose ("Landau scaling"):

$$\overrightarrow{A} = (0, Bx, 0)$$

$$\overrightarrow{B} = \nabla \times A = (0, 0, B)$$

Notice that the degree of freedom of the movement along axis Z doesn't depend on both degrees of freedom of the orbital movement in the plane vertical to axis Z. The self states of the hamiltonian are

$$\hat{H}|p_2 nr\sigma_2 >= E_{p_2 nr\sigma_2}|p_2 nr\sigma_2 >$$

$$E_{p_2 nr\sigma_2} = \frac{p_2^2}{2m} + \left(\frac{1}{2} + n\right)w_c - \gamma B\sigma_2$$

$$w_c \equiv \frac{eB}{m}, -\infty \leq p_2 \leq \infty$$

$$n = 0, 1, 2..., \sigma_2 = \pm 1$$

$$r = 1, 2...\frac{L^2}{2\pi}eB \text{ (Landau's direction)}$$

Calculate the distribution function $Z(\beta)$ for the electron gas, assuming that it's possible to relate to the electrons like Boltzman's particles.

$$Z_N = \frac{1}{N!}Z_1^N$$

From here, derive expressions for the magnetization $\tilde{M}$ and for the susceptibility χ. Notice the expressions you got includes two terms

(a) Diamagnetic term because of the "orbital" interaction of the electron with the field.

b Paramagnetic term because of the interaction of the electron spin with the field.

Diamagnetism is a quantal effect. To prove this argument, take the hamiltonian $\hat{H}$ omitting the spin's degree of freedom and calculate the distribution function by the definition of the classical statistical mechanics.

$$Z_1(\beta, B) \equiv \int \frac{d^3xd^2p}{(2\pi)^3} e^{-\beta} H(\vec{x}, \vec{p})$$

Exercise appendix

This appendix includes a simple way to get to the expression for the self energies of the Hamiltonian $\hat{H}$ given in the exercise.

$$\hat{H} = H\left(\hat{Z}\hat{P}_2; \hat{y}\hat{p}_y, \hat{X}\hat{P}_X; \hat{\sigma}_2\right)$$

We'll define a new set of coordinates

$$\hat{\Pi}_x = m\frac{d\hat{x}}{dt} = mi[\hat{H}, \hat{x}] = \hat{p}_x - e\hat{A}_x$$

$$\hat{\Pi}_y = m\frac{d\hat{y}}{dt} = mi[\hat{H}, \hat{y}] = \hat{p}_y - e\hat{A}_y$$

$$\hat{X} \equiv \hat{X} + \frac{1}{mw_2}\hat{\Pi}_y$$

$$\hat{Y} \equiv \hat{y} + \frac{1}{mw_2}\hat{\Pi}_x$$

Classical $\Pi_x\Pi_y$ is the mechanical momentum of the electron, and XY are the expressions for the center of the helix.
note: $[\hat{\Pi}_x, \hat{\Pi}_y] = imw_2$

$$\left[\hat{X}, \hat{Y}\right] = -i\frac{1}{mw_2}$$

X, Y commute with $\hat{\pi}_x, \hat{\pi}_y$, therefore, $\frac{1}{mw_c}\hat{\pi}_y$ is the conjugate of $\hat{\pi}_x$ and $mw_c\hat{X}$ is the conjugate of $\hat{Y}$.
We'll write the hamiltonian through the canonical new coordinates.

$$\hat{H} = H\left(\hat{z}, \hat{p}_z; \hat{\pi}_x, \frac{1}{mw_c}\hat{\pi}_y; \hat{Y}, mw_c\hat{X}; \hat{\sigma}_z\right) =$$

$$= \frac{\hat{p}_2^2}{2m} + \left(\frac{\hat{\pi}_2^2}{2m} + \frac{1}{2}mw_c^2\left(\frac{\hat{\pi}}{mw_c}\right)^2\right) - \gamma B\hat{\sigma}_z$$

notice that the freedom degree $\hat{X}\hat{Y}$ doesn't appear in $\hat{H}$ (as expected) because $\hat{X}\hat{Y}$ are movement constants.
immediately we see that $\hat{H}$ includes transeltore freedom degree (direction z), harmonically freedom degree, spin's freedom degree and degenerated $\hat{X}\hat{Y}$ freedom degree. There for the self energies of the hamiltonian are:

$$E_{p_2nr\sigma_1} = \frac{p_2^2}{2m} + \left(\frac{1}{2} + n\right)w_c - rB\sigma_z$$

It's possible to set Landau's degeneration $r = 1, 2 \ldots \frac{L^2}{2\pi}eB$ from the semy classical equation

$$\mathcal{N} = \int\int \frac{d(mw_cX)\,dY}{2\pi}$$