headingE2170: Polarization of two-spheres system inside a tube

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The problem:

Given two balls in a very long, hollow tube, with length L. The mass of each ball is m, The charge of one ball is $-q$ and the charge of the other one is $+q$. The ball's radius is negligible, and the electrostatic attraction between the balls is also negligible. The balls are rigid and can't pass through each other. The balls are attached to a drop, whose surface tension causes it's gravity constant γ to work on the balls toward each other (The force does not depend on the distance between the balls). The system is in an external electric uniform field $\bar{\varepsilon} = \varepsilon\hat{x}$ and in thermal equilibrium in temperature T.

(a) Write the hamiltonian of the system $H(p_1, p_2, x_1, x_2) = E_k + V(x)$ when E_k is the kinetic energy. Define properly $V(x)$ when $x = x_2 - x_1$ an–d write a diagram of $V(x)$.

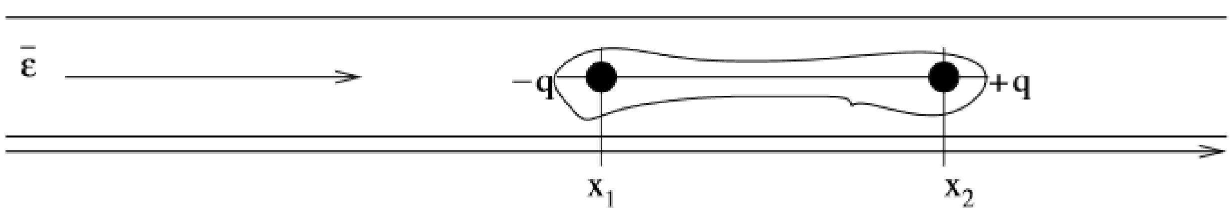
(b) Calculate the partition function $Z(\beta, \varepsilon)$.

(c) Find the probability function of x, $\rho(x)$ and the average distance $\langle x \rangle$ between the balls. Express again $\rho(x)$ by $\langle x \rangle$.

(d) Find the polarization p as a function of ε. Use the partition function.

(e) Develop $P(\varepsilon)$ up to first order in the field: $P(\varepsilon) = P_0 + \chi\varepsilon + O(\varepsilon^2)$.

This development is valid in a weak field, Define what is a weak field. Express your answers with $L, \mathsf{m}, q, \gamma, T, \varepsilon$.



The solution:

(a) Since we assume $L \gg \langle x \rangle$ and we derive in the continuing that $\langle x \rangle = \frac{1}{\beta(\gamma - q\varepsilon)}$, this derivation is valid in a weak field with respect to the mechanical attraction force γ,

$$\varepsilon \ll \gamma/q \tag{1}$$

The hamiltonian of the system is:

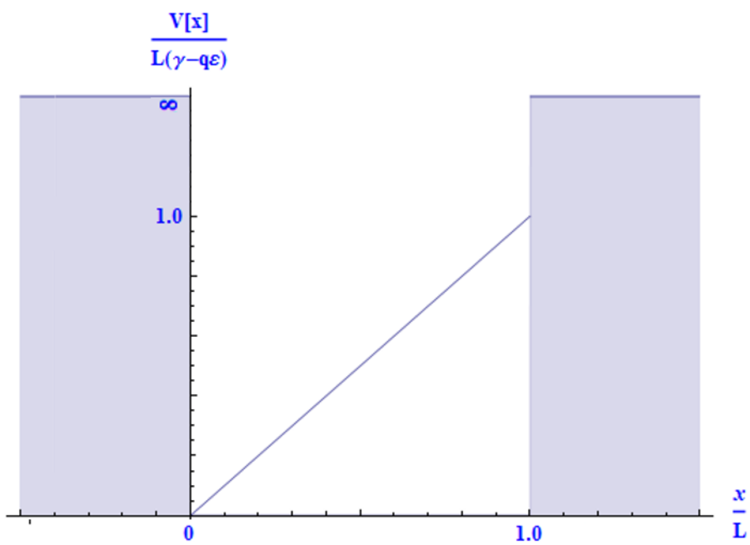
$$\mathcal{H} = E_k + V(x) \tag{2}$$

Where the kinetic energy is:

$$E_k = \frac{p_1^2}{2\text{m}} + \frac{p_2^2}{2\text{m}} \tag{3}$$

The electrostatic potential is $q\varepsilon\left(x_1 - x_2\right)$. The surface tension potential is $\gamma\left(x_2 - x_1\right)$. Thus, the potential energy is,

$$V(x) = \begin{cases} \left(\gamma - q\varepsilon\right) x & 0 < x < L \\ \infty & else \end{cases} \tag{4}$$



Graph 1: Potential energy vs. distance between the particles. Both axes are normalized with respect to its maximum values.

(b) Let us define $\lambda_T^{-1} = \sqrt{\frac{mT}{2\pi\hbar^2}}$. Due to the assumption that $L \gg \frac{1}{\beta(\gamma - q\varepsilon)}$, it is a good approximation to integrate the center of mass, X, from 0 to L while calculating the partition function,

$$Z = \frac{1}{\left(2\pi\hbar\right)^2} \int e^{-\beta\left(\frac{p_1^2}{2m} + \frac{p_2^2}{2m}\right)} dp_1\, dp_2 \int_0^L dX \int_0^L e^{-\beta(\gamma - q\varepsilon)x} dx = \lambda_T^{-2} \cdot L \cdot \frac{\left(1 - e^{-\beta(\gamma - q\varepsilon)L}\right)}{\beta\left(\gamma - q\varepsilon\right)} \tag{5}$$

Thus, the partition function is approximately,

$$Z \approx \frac{L}{\lambda_T^2} \frac{1}{\beta\left(\gamma - q\varepsilon\right)} \tag{6}$$

(c) The probability density function of x is,

$$\rho(x) = \frac{e^{-\beta(\gamma - q\varepsilon)x}}{\int_0^L e^{-\beta(\gamma - q\varepsilon)x} dx} = \beta\left(\gamma - q\varepsilon\right) \cdot e^{-\beta(\gamma - q\varepsilon)x} \tag{7}$$

The mean value of x is,

$$\langle x \rangle = \int_0^L x \cdot \beta\left(\gamma - q\varepsilon\right) e^{-\beta(\gamma - q\varepsilon)x}\, dx = -xe^{-\beta(\gamma - q\varepsilon)x}\Big|_0^L + \int_0^L e^{-\beta(\gamma - q\varepsilon)x}\, dx \tag{8}$$

$$= -Le^{-\beta(\gamma - q\varepsilon)L} - \frac{\left(e^{-\beta(\gamma - q\varepsilon)L} - 1\right)}{\beta\left(\gamma - q\varepsilon\right)} \tag{9}$$

Therefore,

$$\langle x \rangle \approx \frac{1}{\beta\left(\gamma - q\varepsilon\right)} \tag{10}$$

Using $\langle x \rangle$, the probability density function is,

$$\rho_{(x)} \approx \frac{1}{\langle x \rangle} \cdot e^{-x/\langle x \rangle} \tag{11}$$

(d) The polarization is,

$$\langle P \rangle = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon} = \frac{1}{\beta}\frac{\partial\left(-\ln\beta\left(\gamma - q\varepsilon\right)\right)}{\partial \varepsilon} = \frac{q}{\beta\left(\gamma - q\varepsilon\right)} = q\langle x \rangle \tag{12}$$

(e) The Taylor expansion of the polarization is,

$$\langle P \rangle = \frac{q}{\beta\gamma\left(1 - \frac{\varepsilon}{(\gamma/q)}\right)} = \frac{q}{\beta\gamma}\left(1 + \frac{\varepsilon}{(\gamma/q)}\right) + O_{(\varepsilon^2)} = \frac{q}{\gamma}T + \left(\frac{q}{\gamma}\right)^2 T\varepsilon + O_{(\varepsilon^2)} \tag{13}$$

Thus,

$$P_0 = \frac{q}{\gamma}T \qquad\qquad \chi = \left(\frac{q}{\gamma}\right)^2 T \tag{14}$$