# E2060: Particle on a ring with electric field

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**The problem:**

Particle of mass m and charge $e$ is free to move on a ring of radius $R$. The ring is located in the $x-y$ plan. The position of the particle on the ring is $x = R\cos(\theta)$ and $y = R\sin(\theta)$. There is an electric field $E$ in the $x$ direction. The temperature is $T$.

(1) Write the Hamiltonian $H(\theta, p)$ of the particle.

(2) Calculate the partition function $Z(\beta, \mathcal{E})$.

(3) Write an expression for the probability distribution $\rho(\theta)$.

(4) Calculate the mean position $\langle x\rangle$ and $\langle y\rangle$.

(5) Write an expression for the probability distribution $\rho(x)$. Attach a schematic plot.

(6) Write an expression for the polarization. Expand it up to first order in $E$, and determine the susceptibility.

Use the following identities:

$$\frac{1}{2\pi}\int_0^{2\pi} \exp(z\cos(\theta))\, d\theta = I_0(z)$$

$$I_0'(z) = I_1(z)$$

$$I_0(z) = 1 + \left(\frac{1}{4}\right) z^2 + \left(\frac{1}{64}\right) z^{4+\dots}$$

**The solution:**

(1) The Hamiltonian is:

$$\mathcal{H} = \frac{P_\theta^2}{2mR^2} - eER\cos\theta$$

(2) The partition function is therefor:

$$Z = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{\beta P_\theta^2}{2mR^2}} dP_\theta \int_0^{2\pi} e^{\beta eER\cos\theta} d\theta = \sqrt{\frac{mR^2}{2\pi\beta}}\frac{2\pi}{2\pi}\int_0^{2\pi} e^{\beta eER\cos\theta} d\theta = \sqrt{\frac{mR^2}{2\pi\beta}} 2\pi I_0(\beta eER)$$

(3) By definition:

$$\rho(\theta) = \int_{-\infty}^{\infty} \rho(\theta, P_\theta)\, dP_\theta = \int_{-\infty}^{\infty} \frac{e^{-\frac{\beta P_\theta^2}{2mR^2} + \beta eER\cos\theta}}{Z} dP_\theta = \frac{\sqrt{\frac{mR^2}{2\pi\beta}} e^{\beta eER\cos\theta}}{\sqrt{\frac{mR^2}{2\pi\beta}} 2\pi I_0(\beta eER)} = \frac{e^{\beta eER\cos\theta}}{2\pi I_0(\beta eER)}$$

(4) The mean $\langle x \rangle$ is given by:

$$< x >=< R\cos\theta >= \int_0^{2\pi} R\cos\theta\rho\left(\theta\right)d\theta = \int_0^{2\pi} R\cos\theta\frac{\mathrm{e}^{\beta eER\cos\theta}}{2\pi I_0(\beta eER)}d\theta = ...$$

$$... = \frac{R}{2\pi I_0(\beta eER)}\int_0^{2\pi}\cos\theta\mathrm{e}^{\beta eER\cos\theta}d\theta = \frac{RI_1(\beta eER)}{I_0(\beta eER)}$$

The mean $\langle y \rangle$ is given similarly:

$$< y >=< R\sin\theta >= \int_0^{2\pi} R\sin\theta\rho\left(\theta\right)d\theta = \int_0^{2\pi} R\sin\theta\frac{\mathrm{e}^{\beta eER\cos\theta}}{2\pi I_0(\beta eER)}d\theta = 0$$

(5) The probability distribution $\rho\left(x\right)$ is given by a simple substitution of random variable. It is important to notice that $\theta(x)$ is not injective so there is a factor of 2:

$$x = R\cos\theta \rightarrow \theta = \arccos\frac{x}{R}$$

$$\rho\left(x\right) = 2\rho\left(\theta(x)\right)\left|\frac{d\theta}{dx}\right| = \frac{\mathrm{e}^{\beta eEx}}{\pi I_0(\beta eER)}\frac{1}{\sqrt{R^2-x^2}}$$

(6) The polarization is the logarithmic derivation of the partition function by the external electric field:

$$P(E) = \frac{1}{\beta}\frac{\partial\ln Z}{\partial E} = \frac{1}{\beta}\frac{\partial}{\partial E}\ln(\sqrt{\frac{mR^2}{2\pi\beta}}2\pi I_0(\beta eER)) = \frac{1}{\beta}\frac{\partial\ln 2\pi I_0(\beta eER))}{\partial E} = eR\frac{I_0^{'}(\beta eER)}{I_0(\beta eER)}$$

By expanding to a Taylor series near $E = 0$:

$$P(E) = eR\frac{\frac{\beta eER}{2} + o(E^3)}{1 + \frac{(\beta eER)^2}{4} + o(E^4)}$$

The linear susceptibility is therefor:

$$\chi^{(1)} = \frac{\beta e^2R^2}{2}$$