

## Ex2065: Classical gas with a general dispersion relation

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### The problem:

Consider a gas of non-interacting particles in a  $d$  dimensional box, with kinetic energy of the form  $E_p = c|p|^s$ , and temperature  $T$ .

- Find the partition function for  $N$  particles in the box.
- Define  $\gamma = 1 + \frac{s}{d}$  and using (a) show that the energy is  $E = \frac{NT}{\gamma-1}$ .
- Show that the entropy is  $S = \frac{N}{\gamma-1} \ln(PV^\gamma) + f(N)$ , and prove that in an adiabatic process  $PV^\gamma = \text{const}$ .
- Show that the heat capacity ratio is  $C_P/C_V = \gamma$ .

### The solution:

- The partition function for a single particle, which is a simple phase space integration over Boltzman's factor:

$$Z_1 = \frac{1}{(2\pi)^d} \int d^d x \int d^d p e^{-\beta c|p|^s} = \frac{V}{(2\pi)^d} \int d\Omega_d \int_0^\infty p^{d-1} e^{-\beta c p^s} dp \quad (1)$$

where we used generalized spherical coordinates to represent the integral, which is in the form of a Gamma function:

$$Z_1 = \frac{V}{(2\pi)^d} \Omega_d \frac{(c\beta)^{-d/s}}{s} \Gamma\left(\frac{d}{s}\right) \quad (2)$$

$\Omega_d$  is the solid angle in the appropriate dimension (e.g.  $\Omega_2 = 2\pi$  or  $\Omega_3 = 4\pi$ ).

Because the gas is classical, the full partition function is just:  $Z = \frac{(Z_1)^N}{N!}$ .

- The average energy:

$$E = -\frac{\partial \ln Z}{\partial \beta} = -N \frac{\partial \ln Z_1}{\partial \beta} + \frac{\partial \ln N!}{\partial \beta} = \frac{N}{\beta} \frac{d}{s} \quad (3)$$

using the definition of  $\gamma$ , the above fraction becomes:

$$E = NT \frac{d}{d(\gamma-1)} = \frac{NT}{\gamma-1} \quad (4)$$

- By equating the two definitions of the Helmholtz function, we can find the entropy:

$$F = E - \frac{S}{\beta} = -\frac{\ln Z}{\beta} \implies S = \beta E + \ln Z \quad (5)$$

using (4), we get the expression:

$$S = \frac{N}{\gamma-1} + N \ln Z_1 - \ln N! = N \ln \left( V \beta^{-d/s} \right) + f(N) \quad (6)$$

all of the constants and  $N$ -dependence was inserted into  $f(N)$ .

Using the fact that any classical non-interacting gas is obeyed by the ideal gas equation  $\beta = N/PV$ , we rewrite (6):

$$S = N \ln \left( V \left( \frac{PV}{N} \right)^{\frac{1}{\gamma-1}} \right) + f(N) = N \ln \left( \left( \frac{PV^\gamma}{N} \right)^{\frac{1}{\gamma-1}} \right) + f(N) \quad (7)$$

and again by implicitly inserting into  $f(N)$ , we get the desired result:

$$S = \frac{N}{\gamma-1} \ln (PV^\gamma) + f(N) \quad (8)$$

In the adiabatic process  $dQ = 0$ , which implies  $S = \text{const.}$  Thus, from (8) we get the condition:

$$S = \frac{N}{\gamma-1} \ln (PV^\gamma) = \text{const.} \implies PV^\gamma = \text{const.} \quad (9)$$

(d) The general form of heat capacity is:

$$C = \frac{dQ}{dT} = T \frac{dS}{dT} \quad (10)$$

we keep the pressure or volume constant for  $C_P$  and  $C_V$  accordingly:

$$C_P = T \frac{dS}{dT} \Big|_{P=\text{const}} = \frac{NT}{\gamma-1} \frac{d \ln (PV^\gamma)}{dT} = \gamma \frac{NT}{\gamma-1} \frac{d \ln V}{dT} = N \frac{\gamma}{\gamma-1} \quad (11)$$

$$C_V = T \frac{dS}{dT} \Big|_{V=\text{const}} = \frac{NT}{\gamma-1} \frac{d \ln P}{dT} = \frac{N}{\gamma-1} \quad (12)$$

in both cases we used the ideal gas equation, in performing the derivative. Finally, we obtain the relation:

$$\frac{C_P}{C_V} = \gamma \quad (13)$$