Ex2046: Gas in a centrifuge

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The problem:

A cylinder of of radius R rotates about its axis with a constant angular velocity Ω. It contains an ideal classical gas of N particles at temperature T.
Note that the Hamiltonian in the rotating frame is $H'(r,p;\Omega) = H(r,p) - \Omega L(r,p)$ where $L(r,p)$ is the angular momentum.
It is conceptually useful to realize that formally the Hamiltonian is the same as that of a charged particle in a magnetic field ("Coriolis force") plus centrifugal potential $V(r)$. Explain how this formal equivalence can be used in order to make a shortcut in the above calculation.

(1) Find the density distribution as a function of the radial distance from the axis.
(2) Write what is the pressure on the walls.

The solution:

First, we would like to rearrange the Hamiltonian in order to simplify the partition function calculation:

$$H = \frac{p^2}{2\mathsf{m}} - \Omega \cdot L = \frac{p^2}{2\mathsf{m}} - \Omega \cdot r \times p = \frac{p^2}{2\mathsf{m}} - p \cdot \Omega \times r = \frac{(p - \mathsf{m}\Omega \times r)^2}{2\mathsf{m}} - \frac{1}{2}\mathsf{m}\left(\Omega \times r\right)^2 \tag{1}$$

One can see that the term $\mathsf{m}\Omega \times r \rightleftharpoons eA(r)$ and that $V(r) = -\frac{1}{2}\mathsf{m}\left(\Omega \times r\right)^2$ which corresponds to a charged particle in electromagnetic field:

$$H = \frac{1}{2\mathsf{m}}(p - eA(r))^2 + V(r)$$

(1) In order to find the density distribution we will find the partition function of a very thin dr layer of gas distant r from the center of the cylinder:

$$Z_1(r) = \int \frac{\mathrm{d}r\mathrm{d}p}{(2\pi)^3}\mathrm{e}^{-\beta\mathcal{H}} = \int \frac{\mathrm{d}r\mathrm{d}p}{(2\pi)^3}\mathrm{e}^{-\beta\left[\frac{1}{2\mathsf{m}}(p-\mathsf{m}\Omega\times r)^2 - \frac{1}{2}\mathsf{m}(\Omega\times r)^2\right]} \tag{2}$$

We will change the variable $p' = p - \mathsf{m}\Omega \times r$ and since we are working in a cylindrical coordinates the integral of equation (2) will become:

$$Z_1(r) = \int \frac{\mathrm{d}p'}{(2\pi)^3}\mathrm{e}^{-\beta\frac{1}{2\mathsf{m}}p'^2} \int_{z=0}^{z=L} \mathrm{d}z \int_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta\, r\mathrm{d}r\, \mathrm{e}^{\frac{1}{2}\beta\mathsf{m}(\Omega r)^2} = \left(\frac{1}{\lambda_T}\right)^3 V\mathrm{e}^{\frac{1}{2}\beta\mathsf{m}(\Omega r)^2} \tag{3}$$

$$Z_N(r) = \frac{Z_1^N}{N!} \quad \text{N equal the number of atoms in the thin layer} \tag{4}$$

Because each thin layer is in contact with the layer around it after equilibrium is archived all the layers have the same chemical potential which we'll calculate next:

$$\mu = \frac{\partial F}{\partial N} = \frac{1}{\beta}(-\ln(Z_1(r)) + \ln(N(r))) = \frac{1}{\beta}\ln\left(\lambda_T^3\frac{N(r)}{V}\mathrm{e}^{-\frac{1}{2}\beta\mathsf{m}(\Omega r)^2}\right) = \frac{1}{\beta}\ln(\left(\lambda_T^3 n(r)\right) - \frac{1}{2}\mathsf{m}(\Omega r)^2$$

$$ (5) $$

If we equate $\mu(r) = \mu(0)$ we can find the distribution of the density:

$$
\begin{aligned}
\mu(r) &= \mu(0) \\
\ln\left(\lambda_T^3 n(r) \mathrm{e}^{-\frac{1}{2}\beta \mathbf{m}(\Omega r)^2}\right) &= \ln\left(\lambda_T^3 n(0) \mathrm{e}^{-\frac{1}{2}\beta \mathbf{m}(\Omega \cdot 0)^2}\right) \\
n(r) &= n(0)\mathrm{e}^{\frac{1}{2}\beta \mathbf{m}(\Omega r)^2}
\end{aligned}
\tag{6}
$$

(2) To find the pressure on the walls of the cylinder we will calculate the force on the walls from the partition function of the entire cylinder, not just a thin layer as we did in the previous section.

$$
Z_1 = \int \frac{\mathrm{d}p'}{(2\pi)^3} \mathrm{e}^{-\beta \frac{1}{2\mathbf{m}} p'^2} \int\limits_{z=0}^{z=L} \mathrm{d}z \int\limits_{\theta=0}^{\theta=2\pi} \mathrm{d}\theta \int\limits_{r=0}^{r=R} \mathrm{d}r\, r \mathrm{e}^{\frac{1}{2}\beta \mathbf{m}(\Omega r)^2} = \left(\frac{1}{\lambda_T}\right)^3 \frac{2\pi L}{\mathbf{m}\Omega^2 \beta} \left(\mathrm{e}^{\frac{1}{2}\beta \mathbf{m}(\Omega R)^2} - 1\right) \tag{7}
$$

$$
Z_N(r) = \frac{Z_1^N}{N!} \quad \text{now N equal the number of atoms in the cylinder} \tag{8}
$$

$$
\langle F \rangle_R = \frac{1}{\beta} \frac{\partial \ln(Z_N)}{\partial R} = N \mathbf{m} \Omega^2 R \frac{1}{1 - \mathrm{e}^{-\frac{1}{2}\beta \mathbf{m}(\Omega R)^2}} \tag{9}
$$

$$
P = \frac{F}{A} = \frac{N \mathbf{m} \Omega^2 R}{2\pi R L} \frac{1}{1 - \mathrm{e}^{-\frac{1}{2}\beta \mathbf{m}(\Omega R)^2}} \tag{10}
$$

We will replace N with $nV = n\pi R^2 L$ and get:

$$
P = \frac{n\pi R^2 L \mathbf{m} \Omega^2 R}{2\pi R L} \frac{1}{1 - \mathrm{e}^{-\frac{1}{2}\beta \mathbf{m}(\Omega R)^2}} = n \frac{\mathbf{m} R^2}{2} \Omega^2 \frac{1}{1 - \mathrm{e}^{-\frac{1}{2}\beta \mathbf{m}(\Omega R)^2}} \tag{11}
$$

If $\Omega \to 0$ we can expand the exponent $\mathrm{e}^x = 1 + x ...$ and get:

$$
P \overset{\Omega \to 0}{\approx} n \frac{\mathbf{m} R^2}{2} \Omega^2 \frac{1}{1 - \left(1 - \frac{1}{2}\beta \mathbf{m}(\Omega R)^2\right)} = \frac{n}{\beta} \tag{12}
$$

Which corresponds to the equation of state of an ideal gas.
On the Other hand if $\Omega \to \infty$ the force on the walls equals the Centripetal force exerted by the wall on all the atoms:

$$
\langle F \rangle_R \overset{\Omega \to \infty}{\approx} N \mathbf{m} \Omega^2 R \tag{13}
$$