Ex 2040: An ideal gas in a box with a gravitation field

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The problem:

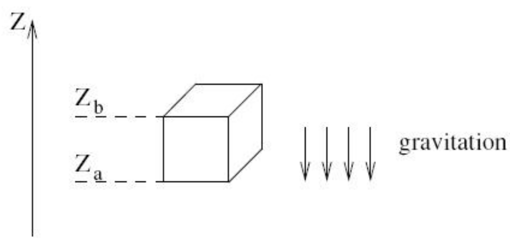
An ideal gas in a 3D box: $V = L \cdot L \cdot (Z_b - Z_a)$ is placed in an external gravitational field that points along $-\hat{z}$.

a)Find the 1-particle partition function $Z_1(\beta, Z_a, Z_b)$.

b)What is the N-particle partition function $Z_N(\beta, Z_a, Z_b)$.

c)What are the forces F_a and F_b acting on the floor and on the ceiling ?

d)What is the difference between these forces? explain your result.



The solution:

a)The one-particle partition function is :

$$Z_1(\beta, Z_a, Z_b) = \iiiint e^{-\beta(\frac{p^2}{2m}+mgz)} \frac{d^3p dx dy dz}{(2\pi)^3} = \frac{L^2}{(2\pi\lambda_T)^3 \beta mg}(e^{-\beta mg Z_a} - e^{-\beta mg Z_b}) \tag{1}$$

Where $\lambda_T = \sqrt{\frac{\beta}{2\pi m}}$ is the thermal wave length.

b)There are no interactions so the partition function gets factorized for each degree of freedom :

$$Z_N(\beta, Z_a, Z_b) = \frac{1}{N!}[Z_1(\beta, Z_a, Z_b)]^N \tag{2}$$

c)We can think of the floor and the ceiling as pistons and we can define the forces F_a and F_b from the work :

$$dW = F_b \cdot dZ_b + F_a(-dZ_a) \tag{3}$$

Which means the work done by the system is positive for lifting the ceiling and for lowering the floor. The forces are :

$$F_b = \frac{1}{\beta}\frac{\partial ln(Z_N)}{\partial Z_b} = Nmg\Big(\frac{e^{-\beta mg Z_b}}{e^{-\beta mg Z_a} - e^{-\beta mg Z_b}}\Big) \tag{4}$$

$$-F_a = \frac{1}{\beta}\frac{\partial ln(Z_N)}{\partial Z_a} = -Nmg\Big(\frac{e^{-\beta mg Z_a}}{e^{-\beta mg Z_a} - e^{-\beta mg Z_b}}\Big) \tag{5}$$

d) The difference is :

$$F_a - F_b = Nmg \tag{6}$$

This result should be intuitive since for $dZ_a = dZ_b = dZ$ we get $-dW = Nmg \cdot dZ$ which is the work done on the system when lifting the box in dZ ,or the gravitational potential energy gained by each particle.