E1825: Adiabatic cooling

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The problem:

Consider an ideal gas whose N atoms have mass m, spin 1/2 and a magnetic moment μ; the energy levels of each particle are $\frac{p^2}{2m} \pm \mu B$ in a magnetic field B where p is the momentum.

(a) Calculate the entropy as $S(T, B) = S_{kin} + S_{spin}$ due to kinetic and spin terms, respectively.

(b) Consider an adiabatic process in which the magnetic field varied from B to zero (adiabatically). Show that the initial T_i and final T_f temperatures are related by the equation:

$$\ln \frac{T_f}{T_i} = \frac{2}{3N}[S_{spin}(T_i, B) - S_{spin}(T_f, 0)]$$

(c) Find the solution for $\frac{T_f}{T_i}$ in the large B limit.

(d) Extend (c) to the case of space dimensionality d and general spin S.

The solution:

(a) The Hamiltonian is:

$$\mathcal{H} = \frac{p^2}{2m} \pm \mu B$$

$$\lambda_T = \frac{h}{\sqrt{2\pi m T}}$$

$$Z = Z_{id} \cdot Z_{spin} = \underbrace{\frac{1}{N!}\left(\frac{V}{\lambda_T^3}\right)^N}_{ideal\ gas} \underbrace{\cdot \left(2\cosh(\beta\mu B)\right)^N}_{spin\ sistem}$$

$$\hookrightarrow F = F_{kin} + F_{spin}$$

$$F \equiv -\frac{1}{\beta}\ln Z(\beta; X)$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = S_{kin} + S_{spin} = N \ln\left(\frac{V}{N\lambda^3}\right) + \frac{5}{2} + N\frac{\partial}{\partial T}\left(T\ln(2\cosh(\beta\mu B)\right)$$

With:

$$S_{kin} = N\ln\left(\frac{V}{N\lambda^3}\right) + \frac{5}{2}$$

$$S_{spin} = N\ln\left(2\cosh(\beta\mu B)\right) - \beta\mu B\tanh\left(\beta\mu B\right)$$

Consider that:

$$\lim_{x\to\infty} \ln(2\cosh x) = \lim_{x\to\infty} \ln(e^x + e^{-x}) = x$$

$$\lim_{x\to\infty} \tanh(x) = 1 \longrightarrow \lim_{x\to\infty} x\tanh(x) = x$$

That's why for $B \to \infty \quad S_{spin} \to 0$ as expected.

And for $B \to 0 \quad S_{spin} = N\ln(2)$.

(b) Adiabatic process:

$$S = S_{kin} + S_{spin} = const$$

$$\Rightarrow S_{kin}(T_i) + S_{spin}(B, T_i) = S_{kin}(T_f) + S_{spin}(0, T_f)$$

From the above we get the desired equation:

$$\Rightarrow N\ln(\frac{\lambda_{T_f}}{\lambda_{T_i}})^3 = \frac{3}{2}N\ln(\frac{T_i}{T_f}) = S_{spin}(0, T_f) - S_{spin}(B, T_i)$$

(c) As $B \to \infty$ $S_{spin}(T_f, B) = 0$

$$\to \ln\left(\frac{T_f}{T_i}\right) = -\frac{2}{3}\ln(2)$$

$$\Rightarrow T_f = \frac{T_i}{2^{2/3}}$$

(d) In d dimantions: $\lambda^3 \to \lambda^d$, spin has $2s+1$ states

$$\frac{d}{2}\ln\left(\frac{T_f}{T_i}\right) = -\ln(2s+1) \to T_f = \frac{T_i}{2s+1}\frac{d}{2}$$