Ex1816: Cooling by adiabatic demagnetization

Submitted by: Zion Hazan and Yair Yifrach

The problem:

Consider a system of N spins on a lattice at temperature T, each spin has a magnetic moment . In presence of an external magnetic field each spin has two energy levels, μH.

1) Evaluate the changes in energy δE and in entropy δS as the magnetic field increases from 0 to H. Derive the magnetization $M(H)$ and show that

$$\delta E = T\delta S - \int_0^H M\left(H'\right) dH'.$$

Interpret this result.

2) What is the temperature change when H is reduced to zero in an adiabatic process. Explain how can this operate as a cooling machine to reach $T \approx 10^{-4}K$. (Note: below $10^{-4}K$ in realistic systems spin-electron or spin-spin interactions reduce $S\left(T, H=0\right) \to 0$ as $T \to 0$. This method is known as cooling by adiabatic demagnetization).

The solution:

1) Let's recall the definition of Helmholtz free energy:

$$F = E - TS \tag{1}$$

Solve for E:

$$E = TS + F \tag{2}$$

Where the energy, entropy and Helmholtz free energy are functions of T, H, N.
Let's differentiate Eq.(2), remembering that the temperature and the number of spin are held fix:

$$dE = TdS + \left(\frac{\partial F}{\partial H}\right)_{T,N} dH \tag{3}$$

Let's consider the average magnetization M as the generalized force, conjugate to the external magnetic field H:

$$M(T,H,N) = -\left(\frac{\partial F}{\partial H}\right)_{T,N} \tag{4}$$

Plug in Eq.(4) to Eq.(3) and integrate over the field:

$$\delta E = T\delta S - \int_0^H M(T,H',N)dH' \tag{5}$$

Where δE and δS are respectively the change of the energy and entropy during the change of the external magnetic field.

2) The Hamiltonian of the interaction of a spin with magnetic moment μ in a magnetic field H is:

$$\mathcal{H} = -\mu_i H \tag{6}$$

Where μ_i is either $+\mu$ or $-\mu$

Let's calculate the partition function of a single spin:

$$Z_1 = \sum_{\mu_i} \exp(\beta\mu_i H) = \exp(\beta\mu H) + \exp(-\beta\mu H) = 2\cosh(\beta\mu H) \tag{7}$$

Thus, since we neglect interaction between the spins and consider only the interaction with the external field, the partition function for N spins will be:

$$Z = Z_1^N = (2\cosh(\beta\mu H))^N \tag{8}$$

In an adiabatic process there is no change in entropy.

The entropy is derived by:

$$\begin{aligned} S = -\frac{\partial F}{\partial T} = \frac{\partial}{\partial T} T ln(Z) = ln(Z) + T\frac{\partial}{\partial T} ln(Z) = \\ = N\left(ln\left(2\cosh\left(\frac{\mu H}{T}\right)\right) - \frac{\mu H}{T}\tanh\left(\frac{\mu H}{T}\right)\right) \end{aligned} \tag{9}$$

Where we used $F = -Tln(Z)$.

Let's take the differential of Eq.(9):

$$0 = dS(T, H, N) = \frac{\partial S}{\partial T}dT + \frac{\partial S}{\partial H}dH \tag{10}$$

Where in the last equation we assume that N is held fix.

After some algebra, utilizing Eq.(9) in Eq.(10), we arrive at:

$$\frac{dT}{T} = \frac{dH}{H} \tag{11}$$

Let's integrate Eq.(11) over the process:

$$\int_{T_i}^{T_f} \frac{dT}{T} = \int_{H_i}^{H_f} \frac{dH}{H} \Rightarrow T_f = \frac{H_f}{H_i} T_i \tag{12}$$

So, we conclude that:

$$\Delta T_{adiabatic} = \frac{\Delta H}{H_i} T_i \tag{13}$$

We can see, from Eq.(12), that as $H_f \to 0; T_f \to 0$.

This can operate as a refrigerator, in analogy to the carnot cycle applied in gas compression. The process is illustrated in the images below.

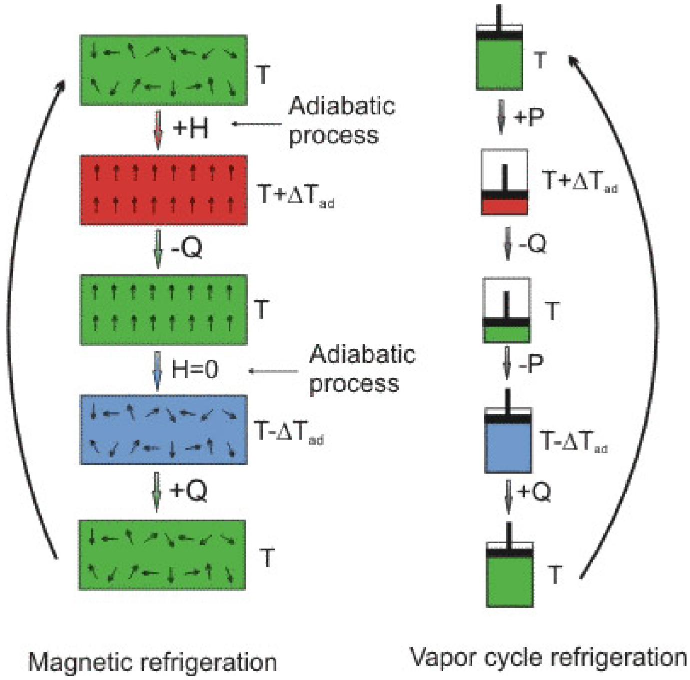

Figure 1: taken from wikipedia

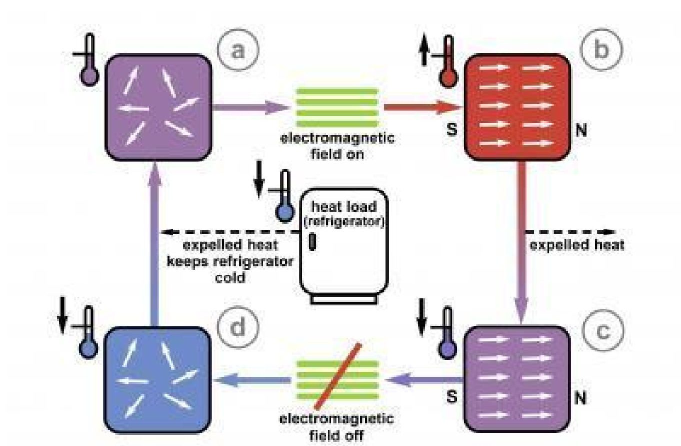

Figure 2: taken from newenergyandfuel.com/

In realistic systems, there also spin-spin and spin-nuclear interactions, so we cannot reduce the effective magnetic field to zero, so there is a limit on how much we can cool the specimen. For further reading see *Thermal Physics, Kittel, SE, page 346*.