Ex1815: Cooling by demagnetization

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The problem:

Consider a solid with N non-magnetic atoms and N_i non-interacting magnetic impurities with spin s. There is a weak spin-phonon interaction which allows energy transfer between the impurities and the non-magnetic atoms.

(a) A magnetic field is applied to the system at a constant temperature T. The field is strong enough to line up the spins completely. What is the change in entropy of the system due to the applied field? (neglect here the spin-phonon interaction).

(b) Now the magnetic field is reduced to zero adiabatically. What is the qualitative effect on the temperature of the solid? Why is the spin-phonon interaction relevant?

(c) Assume that the heat capacity of the solid is $C_V = 3Nk_B$ in the relevant temperature range. What is the temperature change produced by the process (b)? (assume the process is at constant volume).

The solution:

(a) Neglecting the spin-phonon interaction term, the hamiltonian of the system is

$$H = H_s + H_p \tag{1}$$

Where H_s is the contribution of the spins, and H_p is the contribution of the phonons.

However, since the magnetic field does not couple with the phonons, then the change in entropy of the entire system depends only on the spins

$$H_s = -B\sum_{k=1}^{N_i} s_k \tag{2}$$

Where B is an external magnetic field, and s_k can take any value from $-s$ to s

Before the field is turned on, there is no magnetization, and the energy is 0. After the field is turned on, it takes the value B_i which is assumed to be strong enough so that all of the spins are pointing in the same direction. The change in energy is then

$$\Delta E = -B_i N_i s \tag{3}$$

The partition function of the spins is given by

$$Z_s = \left(\sum_{s_1=-s}^{s} e^{\beta B s_1}\right)^{N_i} = \left(\frac{e^{-\beta Bs} - e^{\beta B(s+1)}}{1-e^{\beta B}}\right)^{N_i} = \left(\frac{\sinh\left(\frac{\beta B}{2}(2s+1)\right)}{\sinh\frac{\beta B}{2}}\right)^{N_i} \tag{4}$$

The entropy of the spins is given by

$$S_s = -\left(\frac{\partial F}{\partial T}\right)_B = \frac{\partial}{\partial T}(T\log Z_s) = \log Z_s + \beta E_s \tag{5}$$

Since the field B_i is strong enough to line up all the spins, the following condition is met $\beta B_i \gg 1$ The change in entropy of the entire system is then given by

$$
\begin{aligned}
\Delta S &= S_s(B = B_i) - S_s(B = 0) \\
&= \log Z_s(B = B_i) - \log Z_s(B = 0) + \beta \Delta E \\
&= N_i \log \left(\frac{\sinh\left(\frac{\beta B_i}{2}(2s+1)\right)}{\sinh \frac{\beta B_i}{2}} \right) - N_i \log(2s+1) - \beta N_i B_i s \\
&\approx -N_i \log(2s+1)
\end{aligned}
\tag{6}
$$

This result could generally be exected, given eq. (3).

(b)

- **"What is the qualitative effect on the temperature of the solid?"**
  In this process the field B is reduced from some maximal value B_i to 0 adiabatically, i.e. the total entropy remains constant ($\Delta S = 0$). The field B couples only to the spins (i.e. it does not interact with the phonons, neglecting the spin-phonon interaction term), it is expected therefore that a change in the temperature of the system is determined by a change in entropy of the degrees of freedom of the spins (as the total entropy is fixed). Since the entropy of the spins increases as B decreases, the temperature correspondingly decreases as well (Figure 1). The system therefore is expected to cool down as B is switched off in an adiabatic process.

- **"Why is the spin-phonon interaction relevant?"**
  As the field is reduced from B_i to 0, the entropy of the degrees of freedom of the spins may increase (as the spins become less ordered), therefore a decrease in entropy of the lattice vibrational modes allows for the overall entropy to remain fixed. This process is made possible by the spin-phonon interaction.

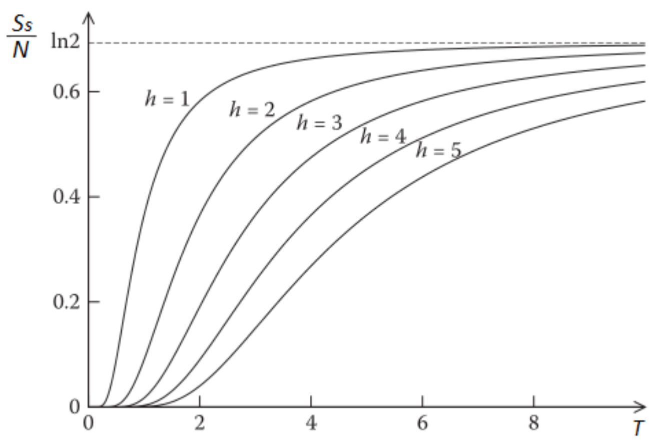

Figure 1: General behavior of the entropy S_s with the temperature T, of a paramagent in a 2-state per single spin system, for different values of the magnetic field $h = \beta B$.

(c) Neglecting the spin-phonon interaction, the total entropy is

$$S = S_s + S_p \tag{7}$$

The heat capacity of the solid during this process is

$$C_V = \left(\frac{\partial E_p}{\partial T}\right)_V = T\left(\frac{\partial S_p}{\partial T}\right)_V = 3N \qquad\qquad (k_B = 1) \tag{8}$$

Since the process is adiabatic, $\delta S = 0$. Combining (7) & (8) gives the following relation

$$\frac{\delta T}{T} = -\frac{1}{3N}\delta S_s \tag{9}$$

Given initial and final temperatures T_i and T_f, integrating both sides

$$\begin{aligned}\log\left(\frac{T_f}{T_i}\right) &= -\frac{1}{3N}\Bigg(S_s(T_f,0) - S_s(T_i,B_i)\Bigg) \\ &= -\frac{1}{3N}\Bigg(\log Z_s(T_f,0) - \log Z_s(T_i,B_i) + \beta_i B_i N_i s\Bigg) \\ &\approx -\frac{N_i}{3N}\log(2s+1)\end{aligned} \tag{10}$$

The final temperature of the process is then

$$T_f \approx T_i\ e^{-\frac{N_i}{3N}\log(2s+1)} \tag{11}$$

Which means that the temperature had dropped, $T_f < T_i$.
The temperature change produced by the process is then

$$\Delta T = T_f - T_i \approx T_i\left[e^{-\frac{N_i}{3N}\log(2s+1)} - 1\right] \tag{12}$$