Ex1814: Adiabatic versus sudden expansion of an ideal gas

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The problem:

N atoms of mass m of an ideal classical gas are in a cylinder with insulating walls, closed at one end by a piston. The initial volume and temperature are V_0 and T_0, respectively.

(a) If the piston is moving out rapidly the atoms cannot perform work, i.e. their energy is constant. Find the condition on the velocity of the piston that justifies this result.

(b) Find the change in temperature, pressure and entropy if the volume increases from V_0 to V_1 under the conditions found in (a).

(c) Find the change in temperature, pressure and entropy if the volume increases from V_0 to V_1 with the piston moving very slowly, i.e. an adiabatic process.

The solution:

(a) According to the equipartition law, in temperature T the kinetic energy of each particle equals $\frac{T}{2}$, so in order to reach the required situation in the question, the velocity of the piston have to be larger than than the thermal velcoity of the particles :

$v_p > v_{Th} = (\frac{T}{m})^{\frac{1}{2}} \Rightarrow v_p > (\frac{T}{m})^{\frac{1}{2}}$

(b) It's given in (a) that the energy of the gas is constant. The energy of an ideal gas is :

$$E = \frac{3NT}{2} \tag{1}$$

Since the energy is constant, we can write :

(*) $E_0 = E_1 \Rightarrow T_0 = T_1$

Therefore :

$$\Delta T = 0 \tag{2}$$

The pressure is :

$$P = \frac{NT}{V} = nT \tag{3}$$

$$\Delta P = P_1 - P_0 = \frac{NT_1}{V_1} - \frac{NT_0}{V_0} \tag{4}$$

We substitute equation (*) in the last equation and we get :

$$\Delta P = \frac{NT_1}{V_1} - \frac{NT_1}{V_0} \tag{5}$$

Since $V_1 > V_0 \Rightarrow P_1 < P_0 \Rightarrow \Delta P < 0$.

$$S = -(\frac{\partial F}{\partial T})_V = N(ln(\frac{1}{n{\lambda_{T_0}}^3}) + \frac{5}{2}) \tag{6}$$

When : $\lambda_T = (\frac{2\pi}{mT})^{\frac{1}{2}}$

N and T are constants so λ_T is also constant and only thing that changes is the volume V.

$$\Delta S = S_1 - S_0 = N(\ln(\frac{1}{n_1{\lambda_T}^3}) + \frac{5}{2}) - N(\ln(\frac{1}{n_0{\lambda_T}^3}) + \frac{5}{2}) = N(\ln(\frac{V_1}{V_0})) \tag{7}$$

Since $V_1 > V_0 \Rightarrow \ln(\frac{V_1}{V_0}) > 0 \Rightarrow \Delta S > 0$.

(c) It's given that the process is adiabatic, i.e $\Delta Q = 0$ and $\Delta S = 0$.

$\Delta S = 0 \Rightarrow S_0 = S_1$.

Because our gas is ideal, we can use equation (7) again (only N is constant) :

$$N(\ln(\frac{1}{n_1{\lambda_{T_1}}^3}) + \frac{5}{2}) = N(\ln(\frac{1}{n_0{\lambda_{T_0}}^3}) + \frac{5}{2}) \quad \Rightarrow T_1 = T_0(\frac{V_0}{V_1})^{\frac{2}{3}} \tag{8}$$

$$\Delta T = T_1 - T_0 = T_0(\frac{V_0}{V_1})^{\frac{2}{3}} - T_0 = T_0((\frac{V_0}{V_1})^{\frac{2}{3}} - 1) \tag{9}$$

In the second transition, we used the expression for T_1 that we got from equation 8.

Since $V_0 < V_1 \Rightarrow \Delta T < 0$.

$$\Delta P = P_1 - P_0 = \frac{NT_1}{V_1} - \frac{NT_0}{V_0} = N(\frac{T_1}{V_1} - \frac{T_0}{V_0}) = \frac{NT_0}{V_0}((\frac{V_0}{V_1})^{\frac{5}{3}} - 1) \tag{10}$$

In the last transition, we used the expression for T_1 that we got from equation 8.

Since $V_0 < V_1 \Rightarrow \Delta P < 0$.