Exercises in Statistical Mechanics

Based on course by Doron Cohen, has to be proofed
Department of Physics, Ben-Gurion University, Beer-Sheva 84105, Israel

This exercises pool is intended for a graduate course in "statistical mechanics". Some of the problems are original, while other were assembled from various undocumented sources. In particular some problems originate from exams that were written by B. Horovitz (BGU), S. Fishman (Technion), and D. Cohen (BGU).

[Exercise 1627]

Equipartition theorem

This is an MCE version of A23: An equipartition type relation is obtained in the following way:
Consider N particles with coordinates $\vec{q}_i$, and conjugate momenta $\vec{p}_i$ (with $i = 1, ..., N$), and subject to a Hamiltonian $\mathcal{H}(\vec{p}_i, \vec{q}_i)$.

(a) Using the classical micro canonical ensemble (MCE) show that the entropy S is invariant under the rescaling $\vec{q}_i \to \lambda \vec{q}_i$ and $\vec{p}_i \to \vec{p}_i/\lambda$ of a pair of conjugate variables, i.e. $S[\mathcal{H}_\lambda]$ is independent of λ, where $\mathcal{H}_\lambda$ is the Hamiltonian obtained after the above rescaling.

(b) Now assume a Hamiltonian of the form $\mathcal{H} = \sum_i \frac{(\vec{p}_i)^2}{2m} + V(\{\vec{q}_i\})$. Use the result that $S[\mathcal{H}_\lambda]$ is independent of λ to prove the virial relation

$$\left\langle \frac{(\vec{p}_1)^2}{m} \right\rangle = \left\langle \frac{\partial V}{\partial \vec{q}_1} \cdot \vec{q}_1 \right\rangle$$

where the brackets denote MCE averages. Hint: S can also be expressed with the accumulated number of states $\Sigma(E)$.

(c) Show that classical equipartition, $\langle x_i \frac{\partial \mathcal{H}}{\partial x_j} \rangle = \delta_{ij} k_B T$, also yields the result (b). Note that this form may fail for quantum systems.

(d) Quantum mechanical version: Write down the expression for the entropy in the quantum case. Show that it is also invariant under the rescalings $\vec{q}_i \to \lambda \vec{q}_i$ and $\vec{p}_i \to \vec{p}_i/\lambda$ where $\vec{p}_i$ and $\vec{q}_i$ are now quantum mechanical operators. (Hint: Use Schrödinger's equation and $\vec{p}_i = -i\hbar\partial/\partial\vec{q}_i$.) Show that the result in (b) is valid also in the quantum case.