E150: The functions $N(E)$ and $Z(T)$ for N spins

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The problem:

Given an N spin system

$$\hat{H} = \sum_{\alpha=1}^{N} \frac{\varepsilon}{2}\hat{\sigma}_z^{(\alpha)}$$

Calculate the partition function $Z_N(\beta)$ in two different ways:

(1) The short way - Calculate $Z_N(\beta)$ by factoring the sum.

(2) The long way - Write the energy levels E_n of the system. Mark with $n = 0$ the basic level and with $n = 1, 2, 3, ...$ the following levels. Find the degeneracy g_n of each level. Use these results to calculate $Z_N(\beta)$.

The solution:

(1) The partition function for a single spin ($E_\downarrow = 0,\ E_\uparrow = \varepsilon$):

$$Z_1(\beta) = 1 + e^{-\beta\varepsilon}$$

For N spins the partition function is given by:

$$Z_N(\beta) = \sum_{\{E^{(\alpha)}\}} e^{-\beta\sum_\alpha E^{(\alpha)}}$$

When α indicates one spin and $\{E^{(\alpha)}\}$ corresponds to a particular configuration of the system. That may seem complicated, but since the spins do not interact we can factorize the sum and get:

$$Z_N(\beta) = \left(1 + e^{-\beta\varepsilon}\right)^N$$

(2) Each spin contributes to the total energy level 0 for spin down and ε for spin up. The total energy for a state with n spins up will than be: $E_n = n\varepsilon$. The degeneracy for such energy level is the number of different groups of n spins up, out of the total N spins in the system. That is also noted as

$$g(n) = \binom{N}{n}$$

the binomial coefficient.
Now, the partition function is a sum over all the states of the system, which can be written as a sum over all the **energy levels**, including their degeneracy. In other words:

$$Z_N(\beta) = \sum_{n=0}^{N} g(n)\, e^{-\beta E_n} = \sum_{n=0}^{N} \binom{N}{n} e^{-\beta n\varepsilon}$$

Which we recognize as Newton's binomial expansion for:

$$Z_N(\beta) = \left(1 + e^{-\beta\varepsilon}\right)^N$$