E060: Oscillator in a microcanonical state

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The problem:

Assume that a harmonic oscillator with freqwency Ω and mass m is prepared in a microcanonical state with energy E.

(1) Write down the probability distribution $\rho(x,p)$.

(2) Find the projected probability distribution $\rho(x)$

The solution:

(1) The Hamiltonian: $H(x,p) = \frac{1}{2m}p^2 + \frac{m\Omega^2}{2}x^2$.
For a microcanonical state with energy E we get:

$$\rho(x,p) = A\delta(H(x,p) - E)$$

where A is a normalization factor we will find later.

(2) The projected probability distribution is given by:

$$\begin{aligned}
\rho(x) &= A\int_{-\infty}^{\infty}\delta(\frac{1}{2m}p^2 + \frac{m\Omega^2}{2}x^2 - E)\frac{dp}{2\pi} \\
&= \frac{A}{\pi}\sqrt{\frac{m}{2}}\int_0^{\infty}\frac{\delta(u + \frac{m\Omega^2}{2}x^2 - E)}{\sqrt{u}}du \\
&= \begin{cases} \frac{A}{\pi\Omega}\frac{1}{\sqrt{\frac{2E}{m\Omega^2} - x^2}} & ; \quad x^2 < \frac{2E}{m\Omega^2} \\ 0 & ; \quad o.w. \end{cases}
\end{aligned}$$

when in the first step we used the simmetricity of p^2 and the change of variables: $p^2 = 2mu$.

Finaly we find A:

$$1 = \int \rho(x)\,dx = \frac{A}{\pi\Omega}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \frac{A}{\pi\Omega}\cdot\pi$$

$$A = \Omega$$