E050: Changing random variables $x = \cos(\theta)$

Submitted by: Elkana Porat

The problem:

Assume that the random phase θ has a uniform distribution over the range $[0, 2\pi]$. Define a new random variable $x = \cos(\theta)$. What is the probability distribution of x ?

The solution:

We use the formula:

$$\rho(x) = \rho(\theta) \left| \frac{d\theta}{dx} \right|$$

For θ we have:

$$\theta = \cos^{-1}(x)$$

$$\frac{d\theta}{dx} = -\frac{1}{\sqrt{1-x^2}} \ ,$$

and the probability distribution:

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi} & ; \quad 0 < \theta < 2\pi \\ 0 & ; \quad o.w. \end{cases}$$

So for the probability distribution of x we have:

$$\rho(x) = \frac{1}{\sqrt{1-x^2}} * \begin{cases} \frac{1}{2\pi} & ; \quad 0 < \cos^{-1}(x) < 2\pi \\ 0 & ; \quad o.w. \end{cases}$$

$$= \begin{cases} \frac{1}{\pi\sqrt{1-x^2}} & ; \quad -1 < x < 1 \\ 0 & ; \quad o.w. \end{cases}$$

When in the last step we add a factor of 2 because of the degeneracy of the solution of $\theta = \cos^{-1}(x)$ in the range $\theta \in [0, 2\pi]$.