Fluctuations in the number of particles

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The problem:

A closed tank of volume V_0 has N_0 particles. "The system" is a subvolume V. The number of particles in V is the random variable N. Define the random variables $\hat{X}_n$ which determines if a certain particle is in the system.

$$X_n = \begin{cases} 1, & \text{the particle is in } V \\ 0, & \text{the particle is not in } V \end{cases}$$

(a) Express $\hat{N}$ using $\hat{X}_n$. By fundamental theorems on adding independent random variables find $\langle N \rangle$, $Var(N)$.

(b) Find the probability function $f(N)$. Using combinatorial considerations and calculate $\langle \hat{L} \rangle$, $Var(L)$.

(c) Assume $|V/V_0 - \frac{1}{2}| << 1$ and treat N as a continuous random variable. Find the probability function $f(N)$.

The solution:

(a) The probability for finding the particle in the subvolume is $p = \frac{V}{V_0}$. As each particle is in ($X_n = 1$) or out ($X_n = 0$) of the subvolume the number of particles is the sum $N = \sum_{n=1}^{N} X_n$. The random variable $\langle \hat{X_n} \rangle = 1 \cdot p + 0 \cdot q$ can also be sumed, as the expectation operator E is linear in the sense that $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$, then we can denote:

$$\langle N \rangle = \sum_{n=1}^{N_0} \langle X_n \rangle = N_0 p \tag{1}$$

To find $Var(N)$ let us find the variance of X_n and use the linearity property $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$. The random variables X_n are not dependant, meaning they have a Covariance of 0. The variance operator of N is then:

$$Var(N) = \sum_{n=1}^{N_0} Var(X_n) = N_0 \left((p) - (p)^2 \right) = N_0 p(1-p) = N_0 pq \tag{2}$$

(b) The number of ways to choose N out of N_0 objects with no importance to the order is $C_{N_0}^{N} = \frac{N_0!}{N!(N_0-N)!}$. So the probability for the number of particles to be N takes the form of binomial distribution:

$$f(N) = \sum_{n=1}^{N_0} C_N^{N_0} p^N q^{N_0 - N} \tag{3}$$

The expectation value and variance can be found by use of the moment generating function of the binomial distribution $M(v) = (pe^v + (1-p))$ and the linearity properties.

$$\langle N\rangle = \sum_{n=1}^{N_0}\langle X_n\rangle = \sum_{n=1}^{N_0}\frac{\partial M(v)}{\partial v}_{|v=0} = N_0 p \tag{4}$$

In a similar way using the second moment (second derivative) we can find the variance:

$$Var(N) = \sum_{n=1}^{N_0} Var(X_n) = \sum_{n=1}^{N_0}\left[\frac{\partial^2 M(v)}{\partial v^2} - \left(\frac{\partial M(v)}{\partial v}\right)^2\right] = N_0 p(1-p) \tag{5}$$

(c) The case when $|V/V_0 - \frac{1}{2}| << 1$ is of particular interest, it is the extremum of $Var(N) = np(1-p)$ as a function of p.
Considering N as a continues random variable is analogues to having a very large number of particles, thus allowing us to make use of the central limit theorem. The probability function of a random variable with $Var(X) = \sigma^2(x)$ and $\langle X\rangle = \bar{X}$ is:

$$f(X) = \frac{1}{\sqrt{2\pi}\sigma} exp\left[-\frac{(X-\langle X\rangle)^2}{2\sigma^2}\right] \tag{6}$$

It is possible to approximate the probability function found in (b) by making use of Stirling's formula $n! \simeq n^n e^{-n}\sqrt{2\pi n}$, which holds for large n. The full derivation is known as the *De Moivre - Laplace theorem* which states:

$$f(N) = \sum_{n=1}^{N_0} C_{N_0}^{N} p^N q^{N_0} \simeq \frac{1}{\sqrt{2\pi N_0 pq}} exp\left[\frac{-(N-N_0 p)^2}{2N_0 pq}\right] \tag{7}$$

Full derivation can be found at http://mathworld.wolfram.com/BinomialDistribution.html (Eq 34 - 65). For the case where $p = V/V_0 = \frac{1}{2}$ we have:

$$f(N) = \frac{1}{\sqrt{\pi\frac{N_0}{2}}} exp\left[\frac{-\left(N-\frac{N_0}{2}\right)^2}{\frac{N_0}{2}}\right] \tag{8}$$