Average length of chain molecule

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The problem:

A molecule can be described as a chain of N monomers, each monomer has the probability p to be positioned horizontally, adding length a to the molecule, otherwise the monomer is of length b. Let L be the molecule's length, define a random variable $\hat{X}_n$ so that:

$$X_n = \begin{cases} a, & \text{the monomer is horizontal} \\ b, & \text{the monomer is vertical} \end{cases}$$

We assume that the ratio $\frac{a}{b}$ is an Irrational number.

(a) Express $\hat{L}$ using $\hat{X}_n$. By fundamental theorems on adding independent random variables find $\langle L \rangle$, $Var(L)$.

(b) Find $f(L) \equiv P(L = na + (N-n)b)$. Using combinatorial considerations and calculate $\langle \hat{L} \rangle$, $Var(L)$.

(c) Define $\sigma_L = \sqrt{Var(L)}$ What is the behavior of $\sigma_L / \langle L \rangle$ as a function of N?

The solution:

(a) As each monomer is of lenth a or b, the lengh of the molecule is the sum $L = \sum_{n=1}^{N} X_n$. The random variable $\langle \hat{X_n} \rangle = a \cdot p + b \cdot q$ where $q = (1-p)$ can also be sumed. The expectation operator E is linear in the sense that $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$, then we can denote :

$$\langle L \rangle = \sum_{n=1}^{N} \langle X_n \rangle = N \langle X_n \rangle = N(ap + bq) \tag{1}$$

To find $Var(L)$ let us find the variance of X_n and use the linearity property $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$. The random variables X_n are not dependant, meaning they have a Covariance of 0. the variance of L is the sum

$$Var(L) = \sum_{n=1}^{N} Var(X_n) = N\left((pa^2 + qb^2) - (pa + qb)^2\right) \tag{2}$$

(b) The number of ways to choose n out of N objects with no importance to the order in which they were chosen is $C_N^n = \frac{N!}{n!(N-n)!}$. So the probablety for a molecule to be of lengh L takes the form of binomial distribution

$$f(L) = P(L = na + (N-n)b) = \sum_{n=1}^{N} C_N^n p^n q^{N-n} (na + (N-n)b) \tag{3}$$

The expectation value and variance can be found by use of the moment generating function of the binomial distribution $M(v) = \left(pe^{av} + qe^{bv}\right)$ and the linearity properties.

$$\langle L\rangle = \sum_{n=1}^{N}\langle X_n\rangle = N\frac{\partial M(v)}{\partial v}\left(pe^{av} + qe^{bv}\right)_{|v=0} = N\left(pa + qb\right) \tag{4}$$

In a similar way using the second moment (second derivative) we can find the variance:

$$Var(L) = \sum_{n=1}^{N} Var(X_n) = \sum_{n=1}^{N}\left[\frac{\partial^2 M(v)}{\partial v^2} - \left(\frac{\partial M(v)}{\partial v}\right)^2\right] = N\left((pa^2 + qb^2) - (pa + qb)^2\right) \tag{5}$$

(c) by inspecting the behavior of the function with respect to N one can see that:

$$\frac{\sigma}{\langle L\rangle} = \frac{\sqrt{N(pa^2 + qb^2 - (pa + qb)^2)}}{N(pa + qb)} \sim \frac{1}{\sqrt{N}} \tag{6}$$