E010: Average distance between two particles in a box

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The problem:

In a one dimensional box with length L, two particles have random positions x_1, x_2 . The particles don't know about each other. The probability function for finding a particle in a specific place in the box is uniform. Let $r = x_1 - x_2$ be the relative location of the particles. Find $\langle \hat{r} \rangle$ and the dispertion relation σ_r:

(1) By using the adding rules for the expectation values and variances of the independent positions.
(2) By calculating the probability function: $f(r)\,dr = P(r < \hat{r} < r + dr)$.

The solution:

(1) The probability function for each particle is given by:

$$f(x) = \begin{cases} 1/L & , \quad 0 < x < L \\ 0 & , \quad o.w. \end{cases}$$

Thus:

$$\begin{aligned} \langle \hat{x}_{1,2} \rangle &= \int_0^L \frac{x dx}{L} = L/2 \\ \langle \hat{x}_{1,2}^2 \rangle &= \int_0^L \frac{x^2 dx}{L} = L^2/3 \\ \sigma_{1,2}^2 &= \langle x_{1,2}^2 \rangle - \langle x_{1,2} \rangle^2 = L^2/12 \end{aligned}$$

Now, the relative location is defined as $\hat{r} = \hat{x}_1 - \hat{x}_2$, so by using the rules for adding expectation values and variances we get:

$$\langle \hat{r} \rangle = \langle \hat{x}_1 \rangle - \langle \hat{x}_2 \rangle = 0$$

$$\sigma_r^2 = \sigma_1^2 + \sigma_2^2 = L^2/6$$

(2) We need to calculate $P(r < \hat{r} < r + dr)$:

$$P(r < \hat{r} < r + dr) = \iint_{r<(x_1-x_2)<r+dr} f(x_1)\, f(x_2)\, dx_1 dx_2 = \frac{1}{L^2} \iint_{\substack{r<(x_1-x_2)<r+dr \\ 0<x_1,x_2<L}} dx_1 dx_2$$

To write the integration limits explicitly we divide to two cases:

(a) <u>$r > 0$</u>: $\quad x_2 + r = x_1 < L \;\Rightarrow\; x_2 < L - r$

$$P(r < \hat{r} < r + dr) = \frac{1}{L^2} \int_0^{L-r} dx_2 \int_{x_2+r}^{x_2+r+dr} dx_1 = \frac{dr}{L^2}(L - r)$$

(b) <u>$r < 0$</u>: $\quad x_2 + r = x_1 > 0 \;\Rightarrow\; x_2 > -r$

$$P(r < \hat{r} < r + dr) = \frac{1}{L^2} \int_{-r}^{L} dx_2 \int_{x_2+r}^{x_2+r+dr} dx_1 = \frac{dr}{L^2}(L + r)$$

Combining the results we get:

$$f\left(r\right)dr=\frac{dr}{L^2}\left(L-\left|r\right|\right)\ ;\ r\in\left[-L,L\right]$$

And finaly we can calculate:

$$\left\langle \hat{r}\right\rangle =\int_{-L}^{L}\left(L-\left|r\right|\right)r\frac{dr}{L^2}=0$$

$$\sigma_r^2=\left\langle \hat{r}^2\right\rangle =2\int_{0}^{L}\left(L-r\right)r^2\frac{dr}{L^2}=L^2/6$$