Entanglement Criteria

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Separable states (reminder)

A pure state $|\psi\rangle$ is called *separable* iff it can be written as $|\psi\rangle = |A\rangle \otimes |B\rangle$ otherwise it is *entangled*.
An example for a pure separable state is $|\uparrow\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle = |00\rangle$.
An examples for pure entangled states are the Bell states that are well known. The singlet state is one of them: $|singlet\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and we will use it later on.
A mixed state ρ is called *separable* (not entangled) iff it can be written as a convex combination of pure product states:

$$\rho = \sum_i p_i |A_i\rangle\langle A_i| \otimes |B_i\rangle\langle B_i| = \sum_i p_i \rho_i^A \otimes \rho_i^B \tag{1}$$

Where $|A_i\rangle$ and $|B_i\rangle$ are state-vectors on the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of subsystems A and B respectively, and $0 \leq p_i \leq 1$, such that $\sum_i p_i = 1$. In general $\langle A_i|A_j\rangle \neq \delta_{i,j}$, $\langle B_i|B_j\rangle \neq \delta_{i,j}$.
An example for a mixed separable state that contains classical correlations, but no quantum correlations, is $\rho = \frac{1}{2}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$.
An example for a mixed entangled state is a Werner state. This state is invariant under the unitary $U \otimes U$, that is satisfy this: $\rho = (U \otimes U)\rho(U^\dagger \otimes U^\dagger)$. We will discus Werner state later.

Operational separability criteria

A pure state has Schmidt rank $r \leq min(dimA, dimB)$ if it can be decomposed as the unique sum:

$$|\psi\rangle = \sum_i^r \sqrt{p_i} |A_i\rangle \otimes |B_i\rangle \tag{2}$$

Where $p_i > 0$, $\sum_i^r p_i = 1$ and $\langle A_i|A_j\rangle = \langle B_i|B_j\rangle = \delta_{i,j}$. Note that p_i are the eigenvalues of the reduced density matrices (ρ^A, ρ^B) that are non-zero. The state $|\psi\rangle$ is separable iff $r = 1$. If all the Schmidt coefficients are non-zero and equal, then the state is said to be maximally entangled.

Peres-Horodecki criterion (positive partial transpose)

The derivation of this separability condition is best done by writing the density matrix elements explicitly, with all their indices

$$\rho_{m\mu,n\nu} = \sum_r p_r (\rho_r^A)_{mn} (\rho_r^B)_{\mu\nu} \tag{3}$$

Latin indices refer to the first subsystem A, Greek indices to the second one B. Note that this matrix have non-negative eigenvalues. The partial transpose of a composite density matrix is given by transposing only one of the subsystems. Thus, the entries of the density matrix that is partially transposed with respect to A are given by

$$\sigma_{m\mu,n\nu} = (\rho^{T_A})_{m\mu,n\nu} \equiv \rho_{n\mu,m\nu} \tag{4}$$

The σ matrix is Hermitian. Assuming that ρ is separable, as any separable state, it can be decomposed according to (1), and it partial transpose is given by

$$\rho_{sep}^{T_A} = \sum_i p_i(|A_i\rangle\langle A_i|)^T \otimes |B_i\rangle\langle B_i| = \sum_i p_i(\rho_i^A)^T \otimes \rho_i^B \tag{5}$$

The transposed matrices $(\rho^A)^T = (\rho^A)^*$ are non-negative matrices with unit trace. It follows that none of the eigenvalues of σ is negative. This is a necessary condition for Eq. (1) to hold. In other words, if ρ is separable, σ have non-negative eigenvalues. The result that $\rho_{sep}^{T_A} \geq 0$ holds also for partial transposition with respect to subsystem B: $\rho^{T_B} = (\rho^{T_A})^T$.
It was shown that for bipartite systems the converse holds only for low-dimensional systems: $2 \otimes 2$ and $2 \otimes 3$. i.e. if $\rho^{T_A}, \rho^{T_B} \geq 0$ then ρ is separable. In this case PPT is a necessary and sufficient condition for separability. For higher dimensions it is only necessary.

Example for PPT: Werner state

A Werner state is a $N \times N$ dimensional bipartite quantum state that is invariant under the unitary $U \otimes U$ for any unitary U.
In our terminology, we consider a pair of $\frac{1}{2}$-spin particles in an impure singlet, consisting of a singlet fraction x and a random fraction $(1-x)$ for impurity:

$$\rho^W = x|singlet\rangle\langle singlet| + \frac{1}{4}(1-x)\mathbf{1} \tag{6}$$

Where $\mathbf{1}$ is the identity matrix. With all the indices, Eq. (6) becomes

$$\rho^W_{m\mu,n\nu} = xS_{m\mu,n\nu} + \frac{1}{4}(1-x)\delta_{m,n}\delta_{\mu,\nu} \tag{7}$$

Particularly our state ρ^W, take the form

$$\rho^W = \frac{1}{4}\begin{pmatrix} 1-x & 0 & 0 & 0 \\ 0 & x+1 & -2x & 0 \\ 0 & -2x & x+1 & 0 \\ 0 & 0 & 0 & 1-x \end{pmatrix} \tag{8}$$

Our recipe is to partially transpose the density matrix with respect to one of the subsystems. The state is defined by (4×4) matrix with 4 (2×2) blocks. Transposed with respect to B, is done by transposing each one of the four blocks, and therefore:

$$(\rho^W)^{T_B} = \frac{1}{4}\begin{pmatrix} 1-x & 0 & 0 & -2x \\ 0 & x+1 & 0 & 0 \\ 0 & 0 & x+1 & 0 \\ -2x & 0 & 0 & 1-x \end{pmatrix} \tag{9}$$

By blocks inspection $(\rho^W)^{T_B}$ has three eigenvalues equal to $(1+x)/4$ and the fourth eigenvalue is $(1-3x)/4$. The last eigenvalue is the lowest. It is positive if $x < \frac{1}{3}$, and the separability criterion is then fulfilled. If the eigenvalue is positive, it is possible to write ρ^W as a mixture of unentangled product states. This necessary condition is also a sufficient one (low-dimension). This result may be compared with other criteria: Bell's inequality holds for $x \leq \frac{1}{\sqrt{2}}$. The singlet state is violating Bell's inequality because $2\sqrt{2} > 2$. In Werner state the weight of the singlet is x, and therefore it have to fulfil $2\sqrt{2}x \leq 2$ in order to hold Bell's inequality.

Reduction criterion

According to the reduction criterion, if ρ is separable then

$$\rho^A \otimes \mathbf{1} - \rho \geq 0 \qquad \textit{and} \qquad \mathbf{1} \otimes \rho^B - \rho \geq 0 \tag{10}$$

Where ρ^i is the reduce density matrix related to the i-subsystem. Like the partial transpose criterion, the reduction criterion is a necessary and sufficient separability condition only for dimensions $2 \otimes 2$ and $2 \otimes 3$, and a necessary condition otherwise.
Substitute $\rho^A = \Sigma_r p_r(\rho_r^A)\mathrm{Tr}(\rho_r^B) = \Sigma_r p_r(\rho_r^A)$ into Equation (10):

$\rho^A \otimes \mathbf{1} - \rho = \Sigma_r p_r(\rho_r^A) \otimes \mathbf{1} - \Sigma_r p_r(\rho_r^A) \otimes (\rho_r^B) = \Sigma_r p_r(\rho_r^A) \otimes [\mathbf{1} - (\rho_r^B)]$ proofs it.

The parenthesis are then non-negative, because $0 \leq \lambda_i \leq 1$ for all (ρ_r^B)s.
In the above example (Werner state) the reduce density matrices are

$$(\rho^W)^A = (\rho^W)^B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\mathbf{1} \tag{11}$$

According to our recipe we calculate for both ρ^A and ρ^B:

$$(\rho^W)^A \otimes \mathbf{1} - \rho = \frac{1}{2}\mathbf{1} - \rho = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 0 \\ 0 & 1-x & -2x & 0 \\ 0 & -2x & 1-x & 0 \\ 0 & 0 & 0 & 1+x \end{pmatrix} \tag{12}$$

We get here the same blocks as in the PPT, and hence the eigenvalues are non-negative for $x < \frac{1}{3}$.

Concurrence

For both pure and mixed quantum states, there are good measures of the degree of entanglement (entanglement of formation, distillable entanglement, relative entropy of entanglement). Concurrence qualifies as an 'entanglement monotone', i.e. $E_X(\rho) \geq 0$. That is $E_X(\rho) = 0$ iff ρ is separable while $E_X(\mathrm{Bell\,State}) = 1$. For the special case of a pair of qubits, the concurrence is well define, as we shall see.
Let us first consider a pure state $|\psi\rangle$ of a pair of qubits. The concurrence $\mathrm{C}(\psi)$ of this state is defined to be

$$\mathrm{C}(\psi) = \mid \langle\psi|\tilde{\psi}\rangle \mid = \mid \langle\psi|(\sigma_y \otimes \sigma_y)|\psi^*\rangle \mid \qquad ; \qquad |\tilde{\psi}\rangle = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} |\psi^*\rangle \tag{13}$$

Where $|\tilde{\psi}\rangle$ is called the 'spin flip' state of $|\psi\rangle$. Here σ_y denote the Pauli matrix $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $|\psi^*\rangle$ is the complex conjugate. The spin flip operation, when applied to a pure product state, takes the

state of each qubit to the orthogonal state.
The connection between concurrence and entanglement is particularly clear if we express the state in the standard basis:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \tag{14}$$

One can show that $|\psi\rangle$ is separable iff $ad = bc$. A measure of entanglement can be taken as the difference between ad and bc. Indeed, this is what concurrence does: $\mathrm{C}(\psi) = 2 \mid ad - bc \mid$.
We can define the concurrence of a mixed state ρ of two qubits to be the average concurrence of an ensemble of pure states representing ρ, minimized over all decompositions of ρ:

$$\mathrm{C}(\rho) = min \sum_j p_j \mathrm{C}(\psi_j) \tag{15}$$

Where $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ and $|\psi_j\rangle$'s are distinct (but not necessarily orthogonal) normalized pure states of the bipartite system. For any pure product state $|\psi\rangle$, $\mathrm{C}(\psi)$ vanishes according to the definition. Consequently, a state ρ is separable iff $\mathrm{C}(\rho) = 0$.
At this point we give, but do not prove, two remarkable facts about concurrence. First, there always exists a decomposition of ρ that achieves the minimum in Eq. (17) with a set of pure states having the same concurrence. Second, one can find an explicit formula for $\mathrm{C}(\rho)$. It is

$$\mathrm{C}(\rho) = max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\} \tag{16}$$

Where the λ_j's are the *square roots* of the eigenvalues of $\rho\tilde{\rho}$ in descending order. Here $\tilde{\rho}$ is the result of applying the spin flip operation to ρ:

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{17}$$

And the complex conjugation is taken in the standard basis. Even though $\rho\tilde{\rho}$ is not necessarily a Hermitian matrix, its eigenvalues are all real and non-negative because it is the product of two non-negative definite matrices.
The generalized definition of concurrence (I-concurrence) for a pure state is as follows:

$$\mathrm{IC}(\psi) = \sqrt{2(1 - \mathrm{Tr}(\rho^A)^2)} \qquad ; \qquad \rho^A = \mathrm{Tr}_B(\rho) \tag{18}$$

This can then be extended to mixed states as in Eq. (17). In advance to measure the concurrence of Werner state (mentioned above), we remember this property: $\sigma_y \otimes \sigma_y$ is unitary and therefore ρ^W remain unchanged, i.e. $\rho^W = \widetilde{\rho^W}$. This implies $\rho^W\widetilde{\rho^W} = (\rho^W)^2$. This matrix have 3 identical eigenvalues equal to $\frac{1}{2}(1-x)^2$ while the fourth is $\frac{1}{2}(1+3x)^2$. Hence, we calculate their square roots:

$$\mathrm{C}(\rho^W) = max\{0, \lambda_1 - 3\lambda_{2,3,4}\} = max\{0, \frac{1}{2}(3x - 1)\} \tag{19}$$

Once again we get separability for $x < \frac{1}{3}$, and max. entanglement at $x = 1$. Moreover, one can see the linear behaviour of the concurrence. Note that the maximal of tangle, define by $\tau(\rho) \equiv (\mathrm{C}(\rho))^2 = [max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}]^2$ of bipartite system is given by $2(n-1)/n$, where $n = min(dimA, dimB)$.