Traffic

The ASEP model

The ASEP (Asymmetric Exclusion Process) model consists of a one dimensional lattice of N cells. Each cell may contain a particle ($n_i = 1$) or be empty ($n_i = 0$). Time is discrete.
The dynamics is defined as follows: at time t, each one of the particles checks its right and left neighbors. If the left cell is empty, it will hop to it with probability q at time *t + 1*, whereas if the right cell is empty it will hop with probability $1 - q$ at time $t + 1$.
At the boundaries (n_1 or n_N) the dynamics is different: if the cell at time t is empty at the left (or respectively right) boundary, a particle will be injected with probability α (or respectively δ) at time *t+1*. If, on the other hand, the left (or respectively right) boundary is occupied at time t, the particle will be removed from the left (respectively right) cell with probability γ (or respectively β), see figure 1.
This model can be seen as a heat transfer model, where each particle carries

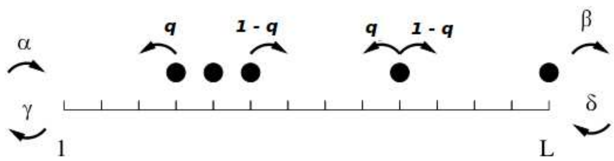

Figure 1: particles in the ASEP model (picture taken from [2])

an energy unit ϵ, and the left and right side remain with constant temperatures $\tau_a,\ \tau_b$ where $exp[-\epsilon/\tau_a] = \alpha/\gamma,\ exp[-\epsilon/\tau_b] = \delta/\beta$. In addition, each boundary has a density of

$$\rho_a = \frac{\alpha}{\alpha + \gamma},\ \rho_b = \frac{\delta}{\delta + \beta}$$

and q is a parameter which determines the external field magnitude (gravitation, electric field). For our purposes we will set $q = \gamma = \delta \to 0,\ \epsilon = 1$, which can be seen as $\tau_a \to 0^+,\ \tau_b \to 0^-,\ \rho_a = 1,\ \rho_b = 0$ and a strong external field.

Traffic

Applying ASEP to traffic we can replace each particle with a car, driving from left to right on a one-lane road. Following this concept, drivers avoid collision by stopping if the next cell is occupied.
The α parameter models the occupancy of the road segment which proceeds the one referred to in our model, whereas the $1 - \beta$ parameter refers to the one following it.
Our aim is to find the current $I(\alpha, \beta)$ in a steady state.
One can solve the above model exactly, however, it is easier to make the approximation of a *random* update process: instead of simultaneously updating the cells, at each time point, an integer $0 \leq i \leq N$ will be chosen with equal probability. The above rules would then apply to the chosen cell (where 0 stands for the case of transition to $i = 1$ with probability α).

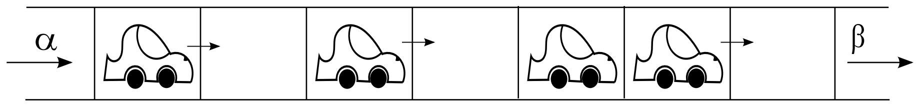

Figure 2: cars in the restricted ASEP model

Steady state and mean-field approximation

For a steady state we will require the average occupancy $\langle n_i(t)\rangle$ to remain unchanged (in time), while the mean field approximation consists of the $n_i's$ to be uncorrelated : $\langle n_i n_j\rangle = \langle n_i\rangle\langle n_j\rangle \equiv p_i p_j \ (i \neq j)$.
Considering one of the cells $2 \leq i \leq N-2$ at time t with occupancy $n_i(t)$, we can write an expression for the occupancy at time $t+1$:

$$
\begin{aligned}
n_i(t+1) = n_i(t) \qquad & \text{with probability } 1-\frac{2}{N+1} \\
n_i(t) + [1-n_i(t)]n_{i-1}(t) \qquad & \text{with probability } \frac{1}{N+1} \\
n_i(t)n_{i+1}(t) \qquad & \text{with probability } \frac{1}{N+1}
\end{aligned}
\tag{1}
$$

The first expression relates to the case where neither i nor $i-1$ are chosen. the second, to the case where $i-1$ is chosen, and the third, to the case where i is chosen.
Averaging (1), we can rewrite it as a rate equation:

$$
\begin{aligned}
\dot{p_i}(t) &\equiv \frac{\langle n_i(t+1)\rangle - \langle n_i(t)\rangle}{1} = -[I_i(t) - I_{i-1}(t)] \\
I_i(t) &= \frac{1}{N+1}\Big(\langle n_i\rangle - \langle n_i n_{i+1}\rangle\Big)
\end{aligned}
\tag{2}
$$

where similar results are found for p_1 and p_N. Using the mean field approximation, the current can be written as

$$I = -p\nabla p \tag{3}$$

Which is different from Fick's law:

$$I = -D\nabla p, \qquad D = const \tag{4}$$

We now invoke the steady state conditions $\dot{p_i}(t) = 0$. Finding $I_i = I_{i-1} = I$, we can write (2) as a recursion with the appropriate boundary conditions:

$$p_{i+1} = 1 - \frac{C}{p_i} \qquad 0 \leq i \leq N \tag{5}$$

$$p_0 = \alpha \tag{6}$$

$$p_{N+1} = 1-\beta \tag{7}$$

where the 'normalized' current $C = (N+1)I$ is our main interest.

Analyzing the flow

Looking at (5) we next try to find the *flow* of the system. That is, the sign of $p_{i+1}-p_i$ near p_i. Defining the *fixed points* to be the solutions of $p_{i+1}(C,p_i)-p_i = 0$ we can characterize the flow according to the fixed points positions.
For $C < 1/4$ there are two fixed points (Figure 3*a*):

$$p_{\pm} = \frac{1}{2}[1 \pm \sqrt{1-4C}] \tag{8}$$

Because of the flow directions p_- is called unstable while p_+ is called stable.
For $C = 1/4$ there is only one fixed point (Figure 3*b*) which is *marginal*, while for $C > 1/4$ there are no real fixed points (Figure 3*c*).

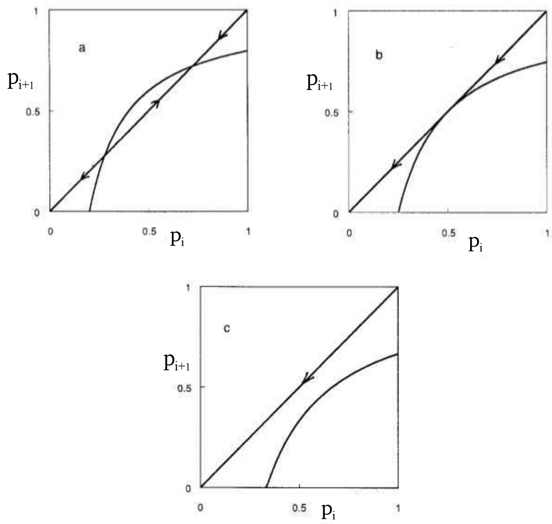

Figure 3: Drawing fixed points and flow direction for various C (picture taken from [1])

Using the above analysis, it can be seen that there are *a priori* four types of solutions:

- *Type 1*: $C < 1/4,\ p_- < p_0 < p_{N+1} < p_+$
- *Type 2*: $C < 1/4,\ p_+ < p_{N+1} < p_0$
- *Type 3*: $C < 1/4,\ p_{N+1} < p_0 < p_-$
- *Type 4*: $C > 1/4,\ p_+ < p_-$

Solutions for the $N \to \infty$ limit

We now find $C(\alpha, \beta)$ for the above solutions as $N \to \infty$:

- *Type 1*: $C < 1/4,\ p_- < p_0 < p_{N+1} < p_+$
  Since the interval $[p_-, p_+]$ is finite, the $p_i's$ must accumulate as $N \to \infty$.

This can be done either at p_- or at p_+. In accordance, *Type 1* solution is split into two possible solutions:
Type 1a: $C < 1/4, \quad p_- < p_0 < p_{N+1}, \quad p_{N+1} = p_+ + 0^-$
Using (6), (7) and the above conditions we get the restrictions for $\alpha,\ \beta$:

$$\alpha > \beta,\ \alpha + \beta < 1$$

And the current

$$C = p_N(1 - p_{N+1}) = \beta(1 - \beta)$$

Type 1b: $C < 1/4, \quad p_0 < p_{N+1} < p_+, \quad p_0 = p_- + 0^+$
Again using (6), (7) we get

$$\alpha < \beta,\ \alpha + \beta < 1$$

And the current

$$C = p_0(1 - p_1) = \alpha(1 - \alpha)$$

- *Type 2*: $C < 1/4, \quad p_+ < p_{N+1} < p_0$
  For this solution to exist as $N \to \infty$, we must have $p_{N+1} = p_+ + 0^+$, otherwise we can find i such that $p_i > 1$ (in fact it will be true for any $p_j,\ j < i$).
  The conditions above and (6), (7) results in the restrictions for α, β:

  $$\alpha + \beta > 1, \beta < 1/2$$

  The current is

  $$C = p_N(1 - p_{N+1}) = \beta(1 - \beta)$$

- *Type 3*: $C < 1/4, \quad p_{N+1} < p_0 < p_-$
  For a similar reasoning to the above, we require $p_0 = p_- + 0^-$. Again using (6), (7) we get:

  $$\alpha < 1/2,\ \alpha + \beta > 1$$

  The current is

  $$C = p_0(1 - p_1) = \alpha(1 - \alpha)$$

- *Type 4*: $C > 1/4, \quad p_+ < p_-$
  For this case we must have $C = 1/4 + 0^+$ and $p_0 \geq 1/2, p_{N+1} \leq 1/2$. otherwise we will have either $p_i > 1$ or $p_i < 0$ for some i. Using this conditions we find

  $$\alpha \geq 1/2,\ \beta \geq 1/2$$

  and

  $$C = 1/4$$

Collecting all of the results, it is possible to draw the phase diagram (Figure 4).

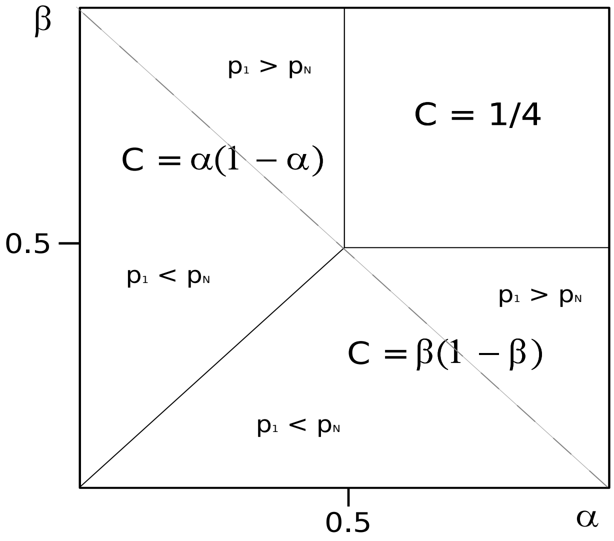

Figure 4: Phase diagram

References

[1] B. Derrida, E. Domany and D. Mukamel, J. Stat. Phys., 69, 667 (1992)

[2] B. Derrida, *Non equilibrium steady states: fluctuations and large deviations of the density and of the current*, arXiv:0703762