Quantum Teleportation

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The Quantum No Cloning Theorem

After performing a quantum computation, one might want to copy the resulting qubit. There are several reasons to clone qubits, for example with many identical qubits many measurements are possible, perhaps enough to obtain the original qubit state.
Let us assume such cloning process is possible. A cloning operator $\hat{U}$ operating on a system with one source qubit χ_1 and one empty target qubit ϕ combined to $|\chi_1, \phi\rangle$, will produce a system with two identical qubits:

$$\hat{U}\,|\chi_1, \phi\rangle = |\chi_1, \chi_1\rangle \tag{1}$$

The cloning operator $\hat{U}$ must be unitary, to preserve normalization. Obviously, such operator should not depend on the qubit it operates on, because if prior knowledge on the qubit state is necessary, that is no cloning. The cloning must work in the same way for a different source qubit:

$$\hat{U}\,|\chi_2, \phi\rangle = |\chi_2, \chi_2\rangle \tag{2}$$

Multiplying (1) by the adjoint of (2) gives:

$$\begin{aligned}\langle\chi_2, \phi|\,\hat{U}^\dagger\hat{U}\,|\chi_1, \phi\rangle &= \langle\chi_2, \chi_2|\chi_1, \chi_1\rangle\\ \langle\chi_2|\chi_1\rangle &= \langle\chi_2|\chi_1\rangle^2\\ \langle\chi_2|\chi_1\rangle\left(\langle\chi_2|\chi_1\rangle - 1\right) &= 0\end{aligned}$$

This equation has only the solutions $\langle\chi_2|\chi_1\rangle = 0$ or $\langle\chi_2|\chi_1\rangle = 1$. Obviously our cloning mechanism works only for equal or orthogonal states, and therefore no general state cloning mechanism exists.

Quantum Teleportation

Quantum teleportation is a method to transfer a quantum state (qubit) without breaking the superposition. Assuming that an entangled pair is shared between the sender and the receiver, all it takes to send a qubit is two classical bits, and losing the entanglement.
Initially, Alice has an unknown qubit, and one half of an entangled pair. Bob has the other half of the pair. The qubit may be any kind of qubit, like spin $\frac{1}{2}$, spin 1, or a double well. Abstractly, we will denote the original qubit $|\psi_A\rangle = \lambda|0\rangle + \mu|1\rangle$. The entangled pair will be $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Using the basis:

$$|A_c, A, B\rangle$$

For the control qubit Alice has and wants to send, her qubit from the pair, and Bob's qubit accordingly.

The initial state is:

$$\begin{aligned}|\psi\rangle &= (\lambda\,|0\rangle + \mu\,|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\\ &= \frac{\lambda}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\mu}{\sqrt{2}}(|100\rangle + |111\rangle)\end{aligned}$$

Alice now applies a CNOT gate, using the first qubit as the control qubit, and the second as the target:

$$\text{CNOT}\,|\psi\rangle = \frac{\lambda}{\sqrt{2}}(|000\rangle + |011\rangle) + \frac{\mu}{\sqrt{2}}\,|110\rangle + |101\rangle)$$

Then Alice applies the Hadamard gate on her first qubit:

$$\begin{aligned}
\text{H CNOT}\,|\psi\rangle &= \frac{1}{2}\left[\lambda(|0\rangle + |1\rangle)\otimes(|00\rangle + |11\rangle) + \mu(|0\rangle - |1\rangle)\otimes(|10\rangle + |01\rangle)\right] \\
&= \frac{1}{2}|00\rangle \otimes (\lambda|0\rangle + \mu|1\rangle) \\
&+ \frac{1}{2}|01\rangle \otimes (\mu|0\rangle + \lambda|1\rangle) \\
&+ \frac{1}{2}|10\rangle \otimes (\lambda|0\rangle - \mu|1\rangle) \\
&+ \frac{1}{2}|11\rangle \otimes (-\mu|0\rangle + \lambda|1\rangle)
\end{aligned}$$

Now Alice measures both her qubits, and sends the results to Bob. Depending on the results Alice has sent him, Bob applies a unitary operator:

$$\begin{aligned}
&\text{Alice measures } 00 \rightarrow \text{Bob applies } \hat{I} \\
&\text{Alice measures } 01 \rightarrow \text{Bob applies } \hat{\sigma}_x \\
&\text{Alice measures } 10 \rightarrow \text{Bob applies } \hat{\sigma}_z \\
&\text{Alice measures } 11 \rightarrow \text{Bob applies } i\hat{\sigma}_y
\end{aligned}$$

Bob now possesses a qubit that is identical to the original qubit Alice had, and the teleportation is complete.

Does teleportation contradict the no cloning theorem? No. The original qubit has been destroyed in the process. Does teleportation convey information faster than the speed of light? Well, of course not. Alice must send bob her measurement using a conventional, slower than light communication method. Until this classical information has arrived, no actual quantum data was transferred. Interestingly, the teleportation may be used with different kinds of qubits, allowing us to transform a calculated result qubit to a different kind of qubit (e.g. to transform an Ion Trap qubit, useful for computing, with a photon, which is handier for communication). Alice and Bob can preshare large quantities of entangled states, and later Alice can teleport quantum states to Bob using classical communication only, which is much easier to accomplish than coherent quantum communication.

Entanglement Swapping And The Return Of The SwitchBoard

The teleportation algorithm described above is linear, and therefore can work on a qubit that is entangled with a fourth particle owned by Carol. The qubits of Bob and Carol would be entangled, while the entanglement between Alice and Carol would be destroyed, together with the entanglement between Alice and Bob. This is a crucial step in the way to a 'Quantum Repeater', allowing us to lengthen the distance quantum information can be sent. An interesting and useful application could be a 'Quantum SwitchBoard'. The SwitchBoard could have preshared entangled states with multiple parties. The operator can then transfer the entanglement so that the parties could have a private entangled state.

We shall start with the state:

$$|\psi\rangle = \frac{1}{2}(|0_C 0_{A_c}\rangle + |1_C 1_{A_c}\rangle) \otimes (|0_A 0_B\rangle + |1_A 1_B\rangle)$$

And after following the protocol, the state will be:

$$\begin{aligned} \text{H CNOT}\,|\psi\rangle &= \frac{1}{2}\,|0_{A_c}0_A\rangle \otimes (|0_C0_B\rangle + |1_C1_B\rangle) \\ &+ \frac{1}{2}\,|0_{A_c}1_A\rangle \otimes (|1_C0_B\rangle + |1_C0_B\rangle) \\ &+ \frac{1}{2}\,|1_{A_c}0_A\rangle \otimes (|0_C0_B\rangle - |1_C1_B\rangle) \\ &+ \frac{1}{2}\,|1_{A_c}1_A\rangle \otimes (|0_C1_B\rangle - |1_C0_B\rangle) \end{aligned}$$

Where we had to keep the subscripts because the order has changed. As before, Alice measures her qubits, and sends the information to either Bob or Carol, which will apply a unitary operator according to the same table as before. As a result Bob and Carol will share an entagled bell state.

Bell States

The above discussion used gates to convey the algorithm more visually. Actually, a physicist may use a different but equivalent notation, which I will introduce here for later use in the following section. This notation uses the four Bell states as an orthonormal base. The states are:

$$\begin{aligned} |\beta_{00}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ |\beta_{01}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\ |\beta_{10}\rangle &= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\ |\beta_{11}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}$$

What Alice did earlier was to measure both her qubits in the Bell basis.

Superdense Coding

A converse algorithm uses a single qubit to send two classical bits. Alice and Bob share a pair of entangled qubits, for example $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice applies one of the following operators on her qubit, thus converting their entangled pair to the matching Bell state:

$$\begin{aligned} &\text{Alice applies } \hat{I} && \rightarrow \text{ Bob receives } |\beta_{00}\rangle \\ &\text{Alice applies } \hat{\sigma}_z && \rightarrow \text{ Bob receives } |\beta_{01}\rangle \\ &\text{Alice applies } \hat{\sigma}_x && \rightarrow \text{ Bob receives } |\beta_{10}\rangle \\ &\text{Alice applies } i\hat{\sigma}_y && \rightarrow \text{ Bob receives } |\beta_{11}\rangle \end{aligned}$$

Now, Alice sends her qubit to Bob. Bob measures the pair in the Bell base, and the result he recieves is the two classical bits Alice sent him.
Alice and Bob can preshare a large number of Bell states. Later, when Alice is far away (e.g the Moon, or Bldg.90), she can send qubits that have twice the amount of information then normal bits.

EPR Detector Efficency Loophole

Bell's inequality is often utilized to experimentally show that the universe violates either locality or realism. However, falsifying all other possible explanations to the measurements has proved to

be a real headache for experimental physicists. One such alternative explanation exploits the fact that no detector is 100 percent efficient, and in particular optical measurements have around 5% - 30% efficiency. The claim is that the fair sampling assumption might be wrong due to a "Hidden Variables" related bias.
In order to analyse what happens when the detection efficiency is not 100%, we will use the CHSH inequality with a different correlation function. The original correlation function was:

$$\langle ab \rangle = \sum_{m,m' \in \pm 1} m m' p_{ab}(m, m')$$

Where $p_{ab}(m, m')$ is the probability to measure m (m') spin projection for particle a (b) . The new correlation function will incorparate the fact that some measurements are undetectable, With $m = 0$ meaning an undetected particle.

$$\widetilde{\langle ab \rangle} = \frac{\sum_{m,m' \in -1,0,1} m m' n_{ab}(m, m')}{\sum_{m,m' \in -1,0,1} n_{ab}(m, m')}$$

The sums exclude the completely undetectable case of $m = m' = 0$. $n_{ab}(m, m')$ is the probabilty of measurement including $m = 0$. If we now assume particle detection dependes only on the single detector efficiency η, we will have:

$$\widetilde{\langle ab \rangle} = \frac{\sum_{m,m' \in -1,0,1} m m' n_{ab}(m, m')}{\sum_{m,m' \in -1,0,1} n_{ab}(m, m')} = \frac{\sum_{m,m' \in \pm 1} m m' \eta^2 p_{ab}(m, m')}{\sum_{m,m' \in \pm 1} \eta^2 p_{ab}(m, m') + 2\eta(1 - \eta)} = \langle ab \rangle \frac{\eta}{2 - \eta}$$

Garg and Mermin prove that the CHSH inequality is valid for these correlation functions as well. Therefore, as Dotan has showed, above 83% efficiency there is no fear that some "Hidden Variable" bias could explain the experimental results.

For the last section see:

A. Garg and N. D. Mermin, Phys. Rev. D 35, 3831 (1987).