Bell's inequality and mermin's EPR

Submitted by: Dotan Davidovich

Bell's inequality

Two particles are emitted in opposite directions , in each side there is a device which measure the particle spin in the angles α or γ with respect to some arbitrary axis , the second device on the other side measure anglesβ and γ (same γ). The possible result for α β γ are respectively

$$a = \pm 1 \ , \ b = \pm 1 \ , \ c = \pm 1 \tag{1}$$

we assume casuality , so after emission the particles do not effect each other and so does the measuring devices.

From (1) we can get the following condition :

$$a(b - c) = \pm(1 - bc) \tag{2}$$

If we repeat this experiment many times, each time has its own set of hidden variables denoted by j. We now can write:

$$a_j b_j - a_j c_j = \pm(1 - b_j c_j) \tag{3}$$

Now averaging over all j , we get the following inequalities :

$$\langle ab\rangle - \langle ac\rangle + \langle bc\rangle \leq 1 \ \ , \ \ \langle ab\rangle - \langle ac\rangle - \langle bc\rangle \geq -1 \tag{4}$$

Which are the original bell inequalities.
For a spin 1/2 singlet state Quantum mechanics predicts $\langle ab\rangle = cos(\theta_{ab})$ When θ is the angle between the detectors angles α and β so we get:

$$|cos(\theta_{ab}) - cos(\theta_{ac})| + cos(\theta_{bc}) \leq 1 \tag{5}$$

This inequality is most strongly violated when $\theta_{ab} = 60°$ $\theta_{ac} = 120°$ $\theta_{bc} = 60°$ namely a 60 degrees separation between 3 angles , then (5) becomes :

$$|\frac{1}{2} + \frac{1}{2}| + \frac{1}{2} \leq 1 \rightarrow \frac{3}{2} \leq 1 \tag{6}$$

The CHSH inequality is a generalization of bells , it takes the second angle of the second observer different so we have 4 possible angles and we get:

$$\langle ab\rangle + \langle ad\rangle + \langle bc\rangle - \langle cd\rangle \leq 2 \tag{7}$$

If we go back to only 3 angles then c and d have maximum correlation so in every measurement they are opposite, so we have :

$$d = -c \ \ and \ \ \langle cd\rangle = -1 \tag{8}$$

Inserting into (7) we get bell's original inequality (4).

Detector efficiency loophole

When measuring the correlation between a and b , real detector not always take a measurement, they have an efficiency limit η which is $0 \leq \eta \leq 1$ This makes the inequality (7) depend on the detectors efficiency . If particle detection depends only on the efficiency of the detector η the correlation function $\langle ab \rangle$ becomes:

$$\langle \tilde{ab} \rangle = \frac{\eta}{2-\eta} \langle ab \rangle \tag{9}$$

Inserting into (7) we get:

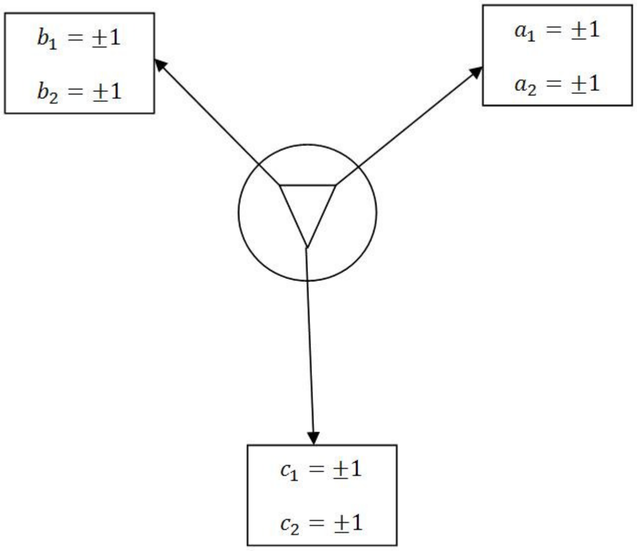
$$\frac{\eta}{2-\eta} |\langle ab \rangle + \langle cb \rangle + \langle ad \rangle - \langle cd \rangle| \leq 2 \rightarrow |\langle ab \rangle + \langle cb \rangle + \langle ad \rangle - \langle cd \rangle| \leq \frac{4}{\eta} - 2 \tag{10}$$

And for the most violating case $\theta = 45°$ we get: $2\sqrt{2} \leq \frac{4}{\eta} - 2$

This will be violated only if $\frac{4}{\eta} - 2 \leq 2\sqrt{2} \quad so \quad \eta \geq \frac{4}{2+2\sqrt{2}} \simeq 0.828$

Mermin EPR

In analogy to the EPR experiment we now consider a 3 particle system. 3 particles are emitted in 3 different directions at each one there is a detector , set to one of two possible setting ,each set can give the answer +1 or -1



While repeating this experiment many times we notice that some setting of all 3 detectors , yield a certain result : for settings $a_1b_2c_2$, $a_2b_1c_2$, $a_2b_2c_1$ we always get an even number of -1 in the detectors. so :

$$a_1b_2c_2 = a_2b_1c_2 = a_2b_2c_1 = 1 \tag{11}$$

Using this result with the LHVT we can predict what will be the result for an $a_1b_1c_1$ setting : taking into account that $a_2^2 = b_2^2 = c_2^2 = 1$ we can write :

$$a_1b_2c_2 \cdot a_2b_1c_2 \cdot a_2b_2c_1 = a_1b_1c_1 = 1 \tag{12}$$

We can now see that the outcome will be, so an even number of detector will measure -1 in this setting too.

This experiment can be modelled by a quantum mechanical state :

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle) \tag{13}$$

And the detectors as: $a_1 = \sigma_x^a \quad a_2 = \sigma_y^a$, $b_1 = \sigma_x^b \quad b_2 = \sigma_y^b$, $c_1 = \sigma_x^c \quad c_2 = \sigma_y^c$.

What would be the possible outcomes for $\sigma_x^a\sigma_y^b\sigma_y^c$ in this state ? (corresponding to a measurement of $a_1b_2c_2$). We remember that:

$$|\uparrow\rangle = \frac{1}{\sqrt{2}}(|x\rangle + |\bar{x}\rangle) = \frac{1}{\sqrt{2}}(|y\rangle + |\bar{y}\rangle) \ , \ |\downarrow\rangle = \frac{1}{\sqrt{2}}(|x\rangle - |\bar{x}\rangle) = \frac{-i}{\sqrt{2}}(|y\rangle - |\bar{y}\rangle) \tag{14}$$

Then :

$$|\psi\rangle = \frac{1}{4}\left[(|x\rangle + |\bar{x}\rangle) \otimes (|y\rangle + |\bar{y}\rangle) \otimes (|y\rangle + |\bar{y}\rangle) + (|x\rangle + |\bar{x}\rangle) \otimes (|y\rangle - |\bar{y}\rangle) \otimes (|y\rangle - |\bar{y}\rangle)\right] \Rightarrow \tag{15}$$

$$|\psi\rangle = \frac{1}{2}(|\bar{x}y\bar{y}\rangle + |\bar{x}\bar{y}y\rangle + |x\bar{y}\bar{y}\rangle + |xyy\rangle$$

Now we see that for measurement setting of $\sigma_x^a\sigma_y^b\sigma_y^c$ we will always get an even number of -1 as assumed in the experiment above. the same can be shown for : $\sigma_y^a\sigma_x^b\sigma_y^c$ and $\sigma_y^a\sigma_y^b\sigma_x^c$.

Now we can look on a different measurement :

$$A|\psi\rangle = (\sigma_x^a\sigma_y^b\sigma_y^c)(\sigma_y^a\sigma_x^b\sigma_y^c)(\sigma_y^a\sigma_y^b\sigma_x^c)|\psi\rangle = |\psi\rangle \tag{16}$$

Under the assumptions of LHVT and using the fact that $(\sigma_y^i)^2 = 1$ we get:

$$(\sigma_x^a\sigma_y^b\sigma_y^c)(\sigma_y^a\sigma_x^b\sigma_y^c)(\sigma_y^a\sigma_y^b\sigma_x^c) = \sigma_x^a\sigma_x^b\sigma_x^c \Rightarrow \sigma_x^a\sigma_x^b\sigma_x^c|\psi\rangle = |\psi\rangle \tag{17}$$

Meaning that a measurement of which will yield the same result as before .

What does QM predicts ?

If we look at $|\psi\rangle$ for $\sigma_x^a\sigma_x^b\sigma_x^c$ we get:

$$|\psi\rangle = \frac{1}{2}(|\bar{x}xx\rangle + |x\bar{x}x\rangle + |xx\bar{x}\rangle + |\bar{x}\bar{x}\bar{x}\rangle \tag{18}$$

And finally :

$$\sigma_x^a\sigma_x^b\sigma_x^c|\psi\rangle = -|\psi\rangle \tag{19}$$

Therefore QM predicts that only an odd number of -1 result will always be measured , in opposite to the LHVT which predicts an even number of -1 will always be measured.

The same result can be obtained from the operator's point of view. First, in order that the measurement (16) will have a meaning all operators must commute. using :

$$\sigma_x^a \sigma_y^a = -\sigma_y^a \sigma_x^a \ \ and \ \ (\sigma_y^i)^2 = 1 \tag{20}$$

The commutations relations are :

$$[\sigma_x^a \sigma_y^b \sigma_y^c, \sigma_y^a \sigma_x^b \sigma_y^c] = \sigma_y^a \sigma_x^b \sigma_y^c \sigma_y^a \sigma_x^b \sigma_y^c - \sigma_y^a \sigma_x^b \sigma_y^c \sigma_x^a \sigma_y^b \sigma_y^c = \sigma_y^a \sigma_x^a \sigma_x^b \sigma_y^b - \sigma_y^a \sigma_x^a \sigma_x^b \sigma_y^b = 0 \tag{21}$$

The same can be shown for the other commutation relations.

What will be the result of this measurement($A|\psi\rangle$)?

Using the commutativity of the a,b,c noted operators and (20) we get :

$$(\sigma_x^a \sigma_y^b \sigma_y^c)(\sigma_y^a \sigma_x^b \sigma_y^c)(\sigma_y^a \sigma_y^b \sigma_x^c) = \sigma_x^a \sigma_y^b \sigma_x^b \sigma_y^b \sigma_x^c = -\sigma_x^a \sigma_y^b \sigma_y^b \sigma_x^b \sigma_x^c = -\sigma_x^a \sigma_x^b \sigma_x^c \tag{22}$$

Therefore :

$$\sigma_x^a \sigma_x^b \sigma_x^c |\psi\rangle = -|\psi\rangle \tag{23}$$

Which is the same contradiction as before.