Spin Echo

Submitted by: Amir Bernat

1 Introduction

Spin Echo is an experimental method used in various quantum set-ups both as a way to elongate coherence time, as well as to measure basic physical phenomena. It is now in use on various fields such as molecular beam experiments, NMR, cold atoms, quantum electronics and the study of crossover from quantum to classical dynamics.

2 Bloch sphere

We can visualize the spin as a vector in the three dimensional "Bloch sphere". In it, the "north pole" is denoted as $|\uparrow\rangle$, the "south pole" as $|\downarrow\rangle$, the "equator" is a superposition of $|\uparrow\rangle$ and $|\downarrow\rangle$ with varying phase differences. In the standard spherical coordinates the angle θ is measured from the north pole to the southern, when using a $\frac{\pi}{2}$ pulse that is the angle by which θ is turned. This angle is given as a Euclidian rotation.
Suppose we prepare the system in the $|\Psi\rangle = |\uparrow\rangle$ state and rotate it by $\frac{\pi}{2}$ about the $\hat{y}$ axis we will get the superposition

$$R\hat{y}(\frac{\pi}{2})|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$$

Suppose we now take the result of the last turn, and create a π pulse, there will be an inversion of $|\uparrow\rangle$ to $|\downarrow\rangle$.
We will use the Bloch sphere later on as a visual aid of the manipulations done on the system at hand.

Bloch sphere of a two site system Image from http://spie.org/x33124.xml?ArticleID=x33124

3 Rabi oscillations

When observing a two site system with energy difference $\epsilon = \mathcal{H}_\uparrow - \mathcal{H}_\downarrow$ and no transition probability.

$$\hat{\mathcal{H}}_0 = \begin{pmatrix} \frac{\epsilon}{2} & 0 \\ 0 & -\frac{\epsilon}{2} \end{pmatrix} = \frac{\epsilon}{2}\sigma_z$$

it is possible to couple these levels by an external oscillating field, this coupling creates a transition probability Δ and the coupled Hamiltonian is denoted as:

$$\hat{\mathcal{H}}_{os} = \begin{pmatrix} 0 & -i\Delta \\ i\Delta & 0 \end{pmatrix} = \Delta\sigma_y$$

The coupling allows transition between the two states which are eigenstates of the unperturbed Hamiltonian. When turning the external filed "on" for a certain amount of time denoted as τ it is possible to divide the initial population between the two sites.
It is common to refer to the time that splits a system that was prepared in one site to an equal superposition of two site as a "$\frac{\pi}{2}$ pulse" with a duration denoted as $\tau_{\frac{\pi}{2}}$.
$\frac{\pi}{2}$ pulse: $\tau_{\frac{\pi}{2}} = \frac{\pi}{2\Omega}$ while $\hat{\mathcal{H}}_{os} = \begin{pmatrix} 0 & -i\Delta \\ i\Delta & 0 \end{pmatrix}$
When the system is completely transferred to the other site it is referred to as a π pulse with duration τ_{π}.
π pulse: $\tau_{\pi} = \frac{\pi}{\Omega}$ while $\hat{\mathcal{H}}_{os} = \begin{pmatrix} 0 & -i\Delta \\ i\Delta & 0 \end{pmatrix}$
A general operator for phase accumulation over time is $U_t = e^{-it\hat{\mathcal{H}}}$, there for we can now assign operators for time propagation without the external oscillating filed, with the external oscillating filed and a duration corresponding to a $\frac{\pi}{2}$ and a π rotations.

$$U_t = e^{-it\hat{\mathcal{H}}_0}$$

$$U_{\frac{\pi}{2}} = e^{-i\tau_{\frac{\pi}{2}}\hat{\mathcal{H}}_{os}}$$

$$U_{\pi} = e^{-i\tau_{\pi}\hat{\mathcal{H}}_{os}}$$

Rabi oscillations are produced by a $\frac{\pi}{2}$ pulse and allowing some time for the system to develop:

$$|\psi(t)\rangle = U_t U_{\frac{\pi}{2}} |\uparrow\rangle = \frac{1}{\sqrt{2}} \left(e^{-i\mathcal{H}_{\uparrow} t} |\uparrow\rangle + e^{-i\mathcal{H}_{\downarrow} t} |\downarrow\rangle \right)$$

It is also possible to use an external filed which has a different frequency - ω then that of the unperturbed Hamiltonian - ϵ, this is called detuning denoted as $2\delta = \epsilon - \omega$. The detuning allows us to control both the amplitude and frequency of the oscillations. In a detuned field the transition frequency is $\Omega = \sqrt{(2\Delta)^2 + \delta^2}$, the switched population will be smaller. When the system is tuned $\delta = 0$ and the population transition can be complete. A general Hamiltonian can be represented as

$$\hat{\mathcal{H}} = \begin{pmatrix} \delta & -i\Delta \\ i\Delta^* & -\delta \end{pmatrix} = \delta\sigma_x + \Delta\sigma_y = \vec{\Omega} \cdot \vec{S}$$

The diagonalization is done by changing to the $|+\rangle$ and $|-\rangle$ base in the precession picture, where we define $\theta_0 = \arctan\left(\frac{2\Delta}{2\delta}\right)$.
The survival probability is then:

$$P_{\uparrow}(t) = |\langle \uparrow |\psi(t)\rangle|^2 = 1 - \sin^2(\theta_0) \cdot \sin^2\left(\frac{\Omega t}{2}\right)$$

4 Ramsey fringes

Ramsey fringes are created by adding another $\frac{\pi}{2}$ pulse some time after a perliminary $\frac{\pi}{2}$ pulse. The fringes are created by a $\frac{\pi}{2} \to t \to \frac{\pi}{2}$ sequence ($t >> \tau_{\frac{\pi}{2}}$). During time t the oscillating field is turned off and so the relative phase of the two sites develops at different rate. When the second $\frac{\pi}{2}$ pulse is turned on transition is once again allowed and so the relative phase between the evolution of the two sites is measured.
The procedure done when using Ramsey's fringes is:

$$\psi(t)\rangle = U_{\frac{\pi}{2}} U_t U_{\frac{\pi}{2}} |\uparrow\rangle = \frac{1}{2}\left(\left(e^{-i\mathcal{H}_\uparrow t} + e^{-i\mathcal{H}_\downarrow t}\right)|\uparrow\rangle + \left(e^{-i\mathcal{H}_\uparrow t} - e^{-i\mathcal{H}_\downarrow t}\right)|\downarrow\rangle\right)$$

The survival probability can now be measured:

$$P_\uparrow(t) = |\langle\uparrow|\psi(t)\rangle|^2 = \left(e^{-i\mathcal{H}_\uparrow t} + e^{-i\mathcal{H}_\downarrow t}\right)\left(e^{i\mathcal{H}_\uparrow t} + e^{i\mathcal{H}_\downarrow t}\right) = \frac{1}{2}\left[1 + \cos((\mathcal{H}_\uparrow - \mathcal{H}_\downarrow)\, t)\right]$$

In the Bloch sphere this will resemble to turning the magnetization vector from the north pole to the equator around the $\hat{y}$ axis to be on the $\hat{x}$ axis, allowing it some time "t" to move on it and then rotating it by $\frac{\pi}{2}$ around the $\hat{y}$ axis. If the time "t" is very close to zero, the magnetization vector will not accumulate any significant phase and can be assumed to have stayed on the $\hat{x}$ axis. After the second $\frac{\pi}{2}$ pulse, it will return to the north pole. If "t" is longer the probability for it to return to the original state will decrease until the magnetization vector will reach the $-\hat{x}$ direction in which there will be a complete population reversal.

5 Spin Echo (Hahn Echo)

Spin echo is a method that is done by introducing another pulse called a "π pulse" which inverts the population of the two sites. Since eventually we inspect the data returned from the inversion this resembles hearing an echo - hence "spin echo".
When using a Spin Echo, the systems will have equal time to evolve before and after the π pulse, thus allowing us to measure the phase difference created in both levels.
A Spin echo sequence is performed by two $\frac{\pi}{2}$ pulses with a π between them. The phase accumulated by the $|\uparrow\rangle$ site is:

$$|\psi(t)\rangle = U_{\frac{\pi}{2}} U_t U_\pi U_t U_{\frac{\pi}{2}} |\uparrow\rangle = \left(e^{-i\mathcal{H}_\uparrow T} e^{-i\mathcal{H}_\downarrow T} + e^{-i\mathcal{H}_\downarrow T} e^{-i\mathcal{H}_\uparrow T}\right)|\uparrow\rangle$$

The calculation of the survival probability is now:

$$P_\uparrow(t) = |\langle\uparrow|\psi(t)\rangle|^2 = \left|\left(e^{-i\mathcal{H}_\uparrow T} e^{-i\mathcal{H}_\downarrow T} + e^{-i\mathcal{H}_\downarrow T} e^{-i\mathcal{H}_\uparrow T}\right)\right|^2 = 1$$

Which means we would expect that there will now be a only probability to be in the $|\uparrow\rangle$ state. On the Bloch sphere it would appear as we have allowed the magnetization vector to progress away from $\hat{x}$ axis along the equator, the reflected it to exactly the same angle from the $\hat{x}$ axis and allowed it the same time so it will return to it's starting point.

6 Perturbations and spatial dependence

Up to now we have considered the different energies created only by the spin states, however, in many experimental set-ups there is an inhomogeneous external field causing perturbations, for example, FORT - far off resonance optical trap which creates different potentials on the $|\uparrow\rangle$ and $|\uparrow\rangle$ states.
The energy difference $\mathcal{H}_\uparrow - \mathcal{H}_\downarrow$ found in Ramsey's fringes survival probability will now depend on another factor describing the measures of stability of the quantum dynamics under perturbations, known as "Quantum fidelity" denoted by $F(t)$. The survival probability is now:

$$P_\uparrow(t) = \frac{1}{2}\left[1 + F(t)\cos((\mathcal{H}_\uparrow - \mathcal{H}_\downarrow)\,T)\right]$$

In the Spin echo experiment the same effect means that we can no longer denote

$$\left|\left(e^{-i\mathcal{H}_\uparrow T}e^{-i\mathcal{H}_\downarrow T} + e^{-i\mathcal{H}_\downarrow T}e^{-i\mathcal{H}_\uparrow T}\right)\right|^2 = 1$$

The location within the external field - $|X\rangle$ acts on the system so $\Psi(x) = |\psi\rangle \otimes |X\rangle$. We will now refer to $|\Psi_\uparrow(x)\rangle$ and to $|\Psi_\downarrow(x)\rangle$, with a new Hamiltonian represented as:

$$\hat{\mathcal{H}} = \begin{pmatrix} \frac{p^2}{2m} + V_\downarrow(x) & \\ & \frac{p^2}{2m} + V_\uparrow(x) \end{pmatrix}$$

The survival probability is now:

$$P_\uparrow(t) = \frac{1}{2}\left[1 + \mathrm{Re}\left(F_{echo}(T)\right)\right]$$

Where $F_{echo}(T) = \langle\Psi_\uparrow(x)|e^{i\hat{\mathcal{H}}_\downarrow T}e^{i\hat{\mathcal{H}}_\uparrow T}e^{-i\hat{\mathcal{H}}_\downarrow T}e^{i\hat{\mathcal{H}}_\uparrow T}|\Psi_\uparrow(x)\rangle$ is the "echo amplitude". When $F_{echo} = 1$ it indicates perfect coherence and $P_\uparrow = 1$, When $F_{echo} = 0$ it indicates complete decoherence and $P_\uparrow = \frac{1}{2}$. This effect allows us to measure the dynamics of the quantum system and external potential studied.

7 References

Decay of Quantum Correlations in Atom Optics Billiards with Chaotic and Mixed Dynamics, M. F. Andersen, A. Kaplan, T. Grunzweig, and N. Davidson http://prl.aps.org/abstract/PRL/v97/i10/e104102